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Abstract	This contribution provides details on the approximations for the expected value of the received information bit rate metric
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Notice	
Release	
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On the Expected Value of the Received Bit Information Rate

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1.0 Purpose

Various expressions for the expected value of received bit information rate (RBIR) are provided.

2.0 Introduction

One of the methods used for the PHY abstraction for system level simulations is based on the received bit information rate (RBIR) [1]. A new formulation based on the RBIR is proposed in [2] to abstract the performance of the maximum likelihood detector in the case of the Rate 2 spatial multiplexing (also referred to as Matrix B for either vertical or horizontal encoding).

Given a received signal,

$$Y = X_n + U \tag{1}$$

where X_n is drawn from a QAM symbol constellation of size *N*, and U is the noise plus interference, then it is shown in [2] that the RIBR is given as

$$RBIR = \frac{1}{\log_2 N}\frac{1}{N}\sum_{n=1}^{N}\int_{-\infty}^{+\infty} p\left(LLR_n\right)\log_2\left\{\frac{N}{1+\exp\left(-LLR_n\right)}\right\}dLLR_n \tag{2}$$

where LLR_n is the log-likelihood ration (LLR) of the *n-th* symbol and $p\left(LLR_n\right)$ is the probability density function of (PDF) LLR_n.

Evaluating the integral in Equation (2) is cumbersome. One approach which is followed in [2] uses look up tables in the evaluation of the RBIR. While the look-table approach works (as shown in [2]), several alternatives are given here that might be computationally simpler.

3.0 Evaluation Methods

In order to evaluate the integral in Equation (2) it is useful to note that this integral is the expected value of a function of a random variable, i.e.

$$RBIR = \frac{1}{\log_2 N}\frac{1}{N}\sum_{n=1}^{N}E\left[g\left(LLR_n\right)\right] \tag{3}$$

where

$$E\left[g\left(LLR_n\right)\right]=\int_{-\infty}^{+\infty}p\left(LLR_n\right)\log_2\left\{\frac{N}{1+\exp\left(-LLR_n\right)}\right\}dLLR_n \quad (4)$$

Thus, computing the RBIR reduces to the evaluation of the expression Equation (4).

3.1 Jensen's Inequality

We now state Jensen's inequality (without proof).

If g is a convex function of a random variable X, then

$$E\left[g\left(X\right)\right]\geq g\left(E\left[X\right]\right) \quad (5)$$

If g is a concave function of a random variable X, then

$$E\left[g\left(X\right)\right]\leq g\left(E\left[X\right]\right) \quad (6)$$

It is easy to show that the function *g* as defined in Equation (4) is concave, thus using Jensen's inequality as shown in Equation (6), the integral in Equation (4) can be upper-bounded by

$$E\left[g\left(LLR_n\right)\right]\leq\log_2\left\{\frac{N}{1+\exp\left(-E\left[LLR_n\right]\right)}\right\} \quad (7)$$

3.2 Using Taylor's Series

In the previous section we considered a bound in the evaluation of Equation (4), in this section a simple derivation for an approximation is given.

The function $g\left(X\right)$ may be expanded in terms of a Taylor's series [3], so that

$$g\left(X\right)=g\left(\mu\right)+\left(X-\mu\right)g'\left(\mu\right)+\frac{\left(X-\mu\right)^2}{2!}g''\left(\mu\right)+\cdots+\frac{\left(X-\mu\right)^k}{k!}g^{(k)}\left(\mu\right)+R_k\left(X\right) \quad (8)$$

where $\mu=E\left[X\right]$, $g^{(k)}\left(X\right)$ is the *k-th* derivative[1] of $g\left(X\right)$ and $R_k\left(X\right)$ is a remainder term that vanishes as *k* get large. Taking the expectation of both side of Equation (8), ignoring $R_k\left(X\right)$, and keeping only the term up to the second derivative, we get

[1] Assuming that the *k-th* derivative exists.

$$E\left[g(X)\right] \approx g(\mu) + \frac{\sigma^2}{2} g''(\mu) \tag{9}$$

Thus, the integral in Equation (4) becomes,

$$\begin{aligned} E\left[g(LLR_n)\right] &= \log_2\left\{\frac{N}{1+\exp\left(-E\left[LLR_n\right]\right)}\right\} \\ &+ \frac{Var(LLR_n)}{2}\frac{d}{dx}\log_2\left\{\frac{N}{1+\exp(-x)}\right\}\Bigg|_{x=E[LLR_n]} \end{aligned} \tag{10}$$

which can be re-written as

$$\begin{aligned} E\left[g(LLR_n)\right] &= \log_2\left\{\frac{N}{1+\exp\left(-E\left[LLR_n\right]\right)}\right\} \\ &+ \frac{Var(LLR_n)}{2\ln(2)}\left\{\frac{\exp\left(-E\left[LLR_n\right]\right)}{1+\exp\left(-E\left[LLR_n\right]\right)}\right\} \end{aligned} \tag{11}$$

where $\ln(2)$ is the natural logarithm of 2.

3.3 Using Differences

In the evaluation of Equation (9) differentiation of the function $g(X)$ is used, instead we can expand $g(X)$ in terms of central differences (Stirling formula), then take the expectation and ignoring terms beyond the second order, yields [4]

$$E\left[g(X)\right] \approx g(\mu) + \frac{1}{2}\frac{g(\mu+h) - 2g(\mu) + g(\mu-h)}{h^2}\sigma^2 \tag{12}$$

The approximation using difference instead of derivates is both easier and might be more accurate. The difference parameter h can now be selected to yield good accuracy for the approximation. Choosing $h = \sqrt{3}\sigma$ is shown to give good accuracy, and in fact is exact for fifth degree polynomials and normally distributed X [5]. In this case, $E\left[g(X)\right]$ becomes

$$E\left[g(X)\right] \approx \frac{2}{3}g(\mu) + \frac{1}{6}g\left(\mu+\sqrt{3}\sigma\right) + \frac{1}{6}g\left(\mu-\sqrt{3}\sigma\right) \tag{13}$$

Thus,

$$E\left[g\left(LLR_n\right)\right] \approx \frac{2}{3}\log_2\left\{\frac{N}{1+\exp\left(-E\left[LLR_n\right]\right)}\right\}$$
$$+\frac{1}{6}\log_2\left\{\frac{N}{1+\exp\left(-E\left[LLR_n\right]-\sqrt{3Var\left(LLR_n\right)}\right)}\right\} \qquad (14)$$
$$+\frac{1}{6}\log_2\left\{\frac{N}{1+\exp\left(-E\left[LLR_n\right]+\sqrt{3Var\left(LLR_n\right)}\right)}\right\}$$

4.0 References

[1] R. Srinivasan et al, “Draft IEEE 802.16m Evaluation Methodology Document,” C80216m-07_080r3.

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[3] A. Papoulis, *Probability, Random Variables, and Stochastic Processes,* 2nd Edition, McGraw-Hill, 1984.

[4] L. Jalloul and J. Holtzman, “Performance Analysis of DS/CDMA with Non-Coherent *M*-ary Orthogonal Modulation in Multipath Fading Channels,” *IEEE J. on Sel. Areas in Commun.*, VOL. 12, NO. 5, pp. 862—870, June 1994.

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