Refractive index profiles and DMD scaling
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Perturbation theory for weakly guiding fibres is described in chapter 18 of "Optical Waveguide Theory" by Allan W. Snyder and John D. Love. On page 376 the basic equations for calculating the change in propagation constant, β, are documented as follows:

$$\left(\beta_p\right)^2 - \left(\beta_u\right)^2 = \frac{k^2 \cdot \int \left[\left(n_p\right)^2 - \left(n_u\right)^2\right] \cdot \left(\psi_u \cdot \psi_p\right) dA}{\int \psi_u \cdot \psi_p \, dA} \qquad (1)$$

where:

- k is the wavenumber,
- n represents the refractive index function,
- ψ is the wavefunction of a mode,
- the subcripts u and p represent the unperturbed and perturbed,
- the integral is over the area of the perturbation.

For first order perturbation it is assumed that $\psi_p = \psi_u$.
If the weak guidance approximations are also assumed as follows:

$$\left(\beta_p\right)^2 - \left(\beta_u\right)^2 \sim 2 \cdot k \cdot n_{co} \cdot \left(\beta_p - \beta_u\right) \qquad (2)$$

$$\left(n_p\right)^2 - \left(n_u\right)^2 \sim 2 \cdot n_{co} \cdot \left(n_p - n_u\right) \qquad (3)$$

where: n_{co} is the peak value of the refractive index of the core of the fibre.

Then we can rewrite equation 1 as:

$$\beta_p = \beta_u + k \cdot \frac{\int \left(n_p - n_u\right) \cdot \left(\psi_u\right)^2 dA}{\int \left(\psi_u\right)^2 dA} \qquad (4)$$

Now the modal delay time, τ, for a length of fibre, L, may be calculated from the following equation:

$$\tau = \frac{L}{c} \cdot \frac{d}{dk} \beta \qquad (5)$$

So that:

$$\tau_p = \tau_u + \frac{L}{c} \cdot \frac{\int \left(n_p - n_u\right) \cdot \left(\psi_u\right)^2 dA}{\int \left(\psi_u\right)^2 dA} \qquad (6)$$

Let the relative delay, RD, of a mode be calculated as:

$$RD = \tau_p - \tau_u \tag{7}$$

Now for the optimum profile τ_u has the same value for all modes.

Therefore:

$$RD = \frac{L}{c} \cdot \frac{\int (n_p - n_u) \cdot (\psi_u)^2 \, dA}{\int (\psi_u)^2 \, dA} \tag{8}$$

Power Law Profiles

For power law profiles it can be shown that:

$$(n_g - n_{g_0}) = \alpha \cdot \left(\frac{g - g_0}{g_1 - g_0} \right) \cdot (n_{g_1} - n_{g_0}) \tag{9}$$

Where:

- g_0 is the optimum power law
- g and g_1 are other non-optimum power laws
- α is a constant.

Therefore, if the perturbation of the refractive index is known for one power law with exponent, g_1, then the perturbation for any other power law with exponent, g, can be calculated. The term, α, is a slight correction term, which in my calculations is assigned values as follows: 1.15 if $g - g_0 > 0$, 0.85 if $g - g_0 < 0$.

Therefore, since:

$$RD_{m,g1} = \frac{L}{c} \cdot \frac{\int (n_{g_1} - n_{g_0}) \cdot (\psi_{m,g_0})^2 \, dA}{\int (\psi_{m,g_0})^2 \, dA} \tag{10}$$

By substitution from equation, 9, we now have:

$$RD_{m,g} = \alpha \cdot \left(\frac{g - g_0}{g_1 - g_0} \right) \cdot \frac{L}{c} \cdot \frac{\int (n_{g_1} - n_{g_0})_{g_1} \cdot (\psi_{m,g_0})^2 \, dA}{\int (\psi_{m,g_0})^2 \, dA} \tag{11}$$

where we have introduced the index, m, to denote a particular mode.

By definition the mean DMD is given by:

$$DMD_{m,g,r} = \sum_m MPD_{m,r} \cdot RD_{m,g} \tag{12}$$

Where:

- $MPD_{m,r}$ is the normalized excited mode power distribution due to the scanning single mode spot.
- The normalization condition is $\sum_m MPD_{m,r}$ equals unity.

After substitution from equation 8 the expression for the DMD becomes:

$$DMD_{m,g,r} = \sum_m MPD_{m,r} \cdot \left[\alpha \cdot \left(\frac{g - g_0}{g_1 - g_0} \right) \right] \cdot RD_{m,g_1} \tag{13}$$

If the DMD is scaled by a factor S we have:

$$S \cdot DMD_{m,g,r} = S \cdot \left[\sum_m MPD_{m,r} \cdot \left[\alpha \cdot \left(\frac{g - g_0}{g_1 - g_0} \right) \right] \cdot RD_{m,g_1} \right] \tag{14}$$

This can be rewriten as:

$$DMD_{m,g_s,r} = \sum_m MPD_{m,r} \cdot \left[\alpha \cdot \left[\frac{(g_s - g_0)}{g_1 - g_0} \right] \right] \cdot RD_{m,g_1} \tag{15}$$

where:

$$g_s = S \cdot (g - g_0) + g_0 \tag{16}$$

and

- g is the unscaled power law exponent,
- g_s is the scaled power law exponent.

Therefore, to first order scaling the DMD by a factor S, scales the refractive index perturbation by the same factor which scales the unscaled power law to the scaled power law according to equation 16.

Central and edge perturbations per the Gigabit Ethernet 81 fibre model

By a similar argument to that of the power law profiles it can be shown that these perturbations scale by the DMD scaling factor S. This should be obvious from inspection of following equation:

$$RD = \frac{L}{c} \cdot \frac{\int (n_p - n_u) \cdot (\psi_u)^2 \, dA}{\int (\psi_u)^2 \, dA}$$

Graphical example of power law scaling

The following graph compares the exact solutions with the scaled solutions based on the perturbation theory described herein. For the scaled graphs an initial power law, g, of 2 was assumed. The relative delays times for g equal to1.85 and 2.1 were then calculated by scaling the relative delays for g equal to 2 using equation 11. Then, as further examples the relative delays for g of 1.85 were scaled by a factor of two and compared with the expected scaled g of 1.73 per equation 16:

$2 \cdot (1.85 - 1.97) + 1.97 = 1.73$

Also, the relative delays for g of 2.1 were scaled by a factor of three and compared with the expected scaled g of 2.36 per equation 16:

$3 \cdot (2.1 - 1.97) + 1.97 = 2.36$

Clearly the approximation is reasonable.