Polarization Mode Dispersion -
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Version 2 - Corrects for an error in eq (5) of 1st Version

Problem - A digital binary fiber optic link has the following specs:

Signaling rate: 10.3125 Gbaud
Fiber: SMF, per IEC 793-2;1992 type B1
Link length: 49 km
Tx laser ctr wavelength: 1550 nm
PMD parameter (D_p): 0.5 ps/sqrt(km)

Calculate the value of time domain pulse spreading, in psec, due to PMD alone, than can occur with a probability of 10^{-12}

In order to answer Vipul's problem, I will utilize a first principles derivation using the PMD impulse response. This is consistent with the approach we have used in the Gb Ethernet spreadsheet model. While I shall only calculate the problem above, this approach can easily be incorporated into the model to give the PMD contribution to the system impulse response and its resulting ISI penalty.

Impulse response for polarization mode dispersion (PMD) in single mode fibers:

The chromatic dispersion contribution to the fiber bandwidth for single mode fiber (SMF) is identical in form to that of the multimode case. For the modal bandwidth contribution, however, the spatial modes of the multimode fiber are replaced by the two orthogonal principal polarization states (PPS) of the SMF, and the difference in signal propagation delay between them is referred to as polarization mode dispersion (PMD). This is normally specified in fiber spec sheets by a PMD dispersion parameter, D_p, with units of psec/km$^{1/2}$, which represents the mean delay difference between the two PPS's for a km of propagation distance. However, the particular polarizations of the PPS's of the fiber are not constant in time, and the actual delay difference between them can be larger or smaller than this with some particular probability distribution. In addition, the amount of input power that couples to each PPS is also non-deterministic, and also described by some suitable probability distribution. We will consider both of these effects simultaneously.

Assume that for a given fiber, the delay difference between the principal polarization states is some time τ_d, and that the power coupled into the two states (normalized to unity) is given by r and (1-r) respectively. The resulting PMD "impulse response" of the fiber, h(t), would be a pair of delta functions with weightings appropriate to their respective powers, ie:

$$h(t) = r\,\delta(t - \tau_d/2) + (1 - r)\,\delta(t + \tau_d/2) \qquad (1a)$$

where the time origin has been chosen half way between the delay times of the two states. Note that h(t) is properly normalized in the sense that:

$$\int_{-\infty}^{\infty} h(t)dt = 1 \qquad (1b)$$

In what follows we will refer to the two principal polarization states which arrive at times $\pm \tau_d / 2$ as PPS(±), respectively.

The rms width of this impulse response, σ_Δ, can be readily calculated:

$$\langle t \rangle = \int_{-\infty}^{\infty} dt\, t\, h(t) = (2r-1)\frac{\tau_d}{2}$$

$$\langle t^2 \rangle = \int_{-\infty}^{\infty} dt\, t^2\, h(t) = \frac{\tau_d^2}{4}$$

$$\rightarrow \sigma_\Delta = \sqrt{\langle t^2 \rangle - \langle t \rangle^2} = \sqrt{r(1-r)}\ \tau_d \qquad (1c)$$

A pulse $p_i(t,\sigma_o)$ launched into a fiber having the above PMD impulse response will result in an output pulse $p_o(t,\sigma_L)$ which is the convolution of the input pulse with the impulse response above, ie:

$$p_o(t,\sigma_L) = p_i(t,\sigma_o) \otimes h(t) \qquad (1d)$$

The parameters σ_o, σ_L are the rms widths of the input and output pulses, respectively. We assume that the input pulse is normalized in the sense of (1b), but is otherwise arbitrary. The rms output width σ_L is obtained from the root mean square sum of the input pulse and PMD impulse response, ie:

$$\sigma_L^2 = \sigma_o^2 + \sigma_\Delta^2 \qquad (2a)$$

as can easily be shown by direct calculation. The pulse spreading is therefore completely determined by the PMD impulse response rms width:

$$\frac{\Delta\sigma}{\sigma_o} = \frac{\sigma_L - \sigma_o}{\sigma_o} = \sqrt{1+\frac{\sigma_\Delta^2}{\sigma_o^2}}\ -1\ \cong\ \frac{1}{2}\left(\frac{\sigma_\Delta}{\sigma_o}\right)^2 \qquad (2b)$$

The solution of our original problem is now reduced to a calculation of the parameter σ_Δ.

Derivation of σ_Δ and its probability distribution:

We wish to derive a probability distribution for σ_Δ. We begin by expressing σ_Δ in terms of an alternate variable, which will make the subsequent mathematics considerably easier. Rather than the variable r, we will use the angle θ between the normalized Stokes vector of the input polarization state and the Stokes vector corresponding to the state PPS(+), ie:

$$\theta = \cos^{-1}(\hat{\mathbf{s}}_i \bullet \hat{\mathbf{s}}_+) \quad [\,0 < \theta < \pi] \qquad (3a)$$

where:

$\hat{\mathbf{s}}_i$ = Stokes vector of input state

$\hat{\mathbf{s}}_+$ = Stokes vector of state PPS(+)

The normalized power r in PPS(+), expressed in terms of this angle, is:

$$r = \cos^2\left(\frac{\theta}{2}\right) \quad [0 < \theta < \pi] \qquad (3b)$$

Using (1c) and (3b), we can re-write σ_Δ as:

$$\sigma_\Delta = \frac{\tau_d}{2}\sin\theta \qquad (3c)$$

NOTE that the projection of the input power on to the state PPS(+) is given by $\cos^2(\theta/2)$, NOT $\cos^2(\theta)$ as for conventional vectors. This is because, on the Poincare Sphere, orthogonal polarization states are on opposite sides of the sphere, so that the angle between them is π (NOT π/2 as for conventional vectors which are orthogonal). This accounts for the factor of two in (3b).

As previously mentioned, there are probability distributions corresponding to the variables τ_d and θ. The former has been determined to obey a Maxwellian distribution of the form:

$$\mathbf{P}(\tau_d) = \sqrt{\frac{2}{\pi}}\left(\frac{\tau_d}{\sigma_d}\right)^2 \frac{e^{-\frac{\tau_d^2}{2\sigma_d^2}}}{\sigma_d} \qquad (4a)$$

Note that $\mathbf{P}(\tau_d)$ is properly normalized, ie:

$$\int_0^\infty d\tau_d\, \mathbf{P}(\tau_d) = 1 \qquad (4b)$$

The connection to D_p, the fiber PMD parameter, is given by the following relation:

$$D_p \sqrt{L} = \langle \tau_d \rangle = \int_0^\infty d\tau_d \, \tau_d \, \mathbf{P}(\tau_d) = \sqrt{\frac{8}{\pi}} \, \sigma_d \qquad (4c)$$

where L is the length of the fiber. This relation indicates that the fiber PMD parameter, D_p, multiplied by the square root of the fiber length is equal to the mean value of the distribution $\mathbf{P}(\tau_d)$. The last equality in (4c) is obtained by explicit integration to calculate the mean. In order to evaluate the probability distribution for θ, consider that the principal states PPS(±) are random for a given fiber, and vary in a random way in time over the Poincare sphere, so the state PPS(+) could be at any point on the Poincare sphere with equal probability. Since polarization states are distributed uniformly on the surface of the sphere, the probability of PPS(+) being contained in any given solid angle is constant. This implies a probability distribution in solid angle $\mathbf{P}(\Omega)$ which is constant, ie:

$$\mathbf{P}(\Omega) d\Omega = \frac{d\Omega}{4\pi} = \frac{1}{4\pi} \sin\theta \, d\theta \, d\phi \equiv \mathbf{P}(\theta, \phi) \, d\theta \, d\phi \qquad (5a)$$

where (θ,ϕ) are the polar angles relative to the direction of the Stokes vector of the input polarization. Integrating $\mathbf{P}(\theta,\phi)$ over ϕ to get $\mathbf{P}(\theta)$:

$$\mathbf{P}(\theta) = \int_0^{2\pi} d\phi \, \mathbf{P}(\theta, \phi) = \frac{1}{2} \sin\theta \qquad (5b)$$

This distribution is properly normalized:

$$\int_0^\pi d\theta \, \mathbf{P}(\theta) = 1 \qquad (5c)$$

We are interested in deriving, from these distributions, a probability distribution $\mathbf{P}(\sigma_\Delta)$ for the parameter σ_Δ. We can do this using the relationship (3c) between σ_Δ, θ, and τ_d. Without going into the details of the derivation, we use (3c) to re-express $\mathbf{P}(\theta)$ in terms of σ_Δ and τ_d. This can be combined with $P(\tau_d)$ to yield a joint probability distribution $\mathbf{P}(\tau_d,\sigma_\Delta)$, given by:

$$\mathbf{P}(\tau_d, \sigma_\Delta) = \sqrt{\frac{2}{\pi}} \left(x \, e^{-\frac{x^2}{2}} \right) \left(\frac{2a}{\sigma_d^2 \sqrt{x^2 - a^2}} \right)$$

where

$$x = \frac{\tau_d}{\sigma_d} \qquad \text{and} \qquad a = \frac{2\sigma_\Delta}{\sigma_d} \qquad (6a)$$

The probability distribution we seek, $\mathbf{P}(\sigma_\Delta)$, is obtained by integrating (6a) over τ_d:

$$P(\sigma_\Delta) = \int_{2\sigma_\Delta}^{\infty} d\tau_d \; \mathbf{P}(\tau_d, \sigma_\Delta) = \frac{2a}{\sigma_d} e^{-\frac{a^2}{2}} \qquad (6b)$$

where a is given in (6a). The distribution $\mathbf{P}(\sigma_\Delta)$ is properly normalized:

$$\int_0^{\infty} d\sigma_\Delta \; \mathbf{P}(\sigma_\Delta) = 1 \qquad (6c)$$

We seek the particular value of σ_Δ, which we denote $\sigma_{\Delta o}$, such that the occurrence of *any* value of $\sigma_\Delta > \sigma_{\Delta o}$ occurs with a probability P_σ equal to 10^{-12}. This condition on $\sigma_{\Delta o}$ can be expressed mathematically as:

$$\int_{\sigma_{\Delta o}}^{\infty} d\sigma_\Delta \; \mathbf{P}(\sigma_\Delta) = P_\sigma \qquad (6d)$$

Substituting (6b) into (6d) and performing the integration yields the following equation for $\sigma_{\Delta o}$:

$$\sigma_{\Delta o} = \frac{\sqrt{\pi}}{4} D_p \sqrt{L} \sqrt{\left|\mathbf{log}_e(P_\sigma)\right|} \qquad (7a)$$

where we have used (4c) to substitute for σ_d. If we assume the initial rms pulse width σ_o is approximately equal to the Baud period B^{-1}, the pulse spreading (2b) is given by:

$$\frac{\Delta\sigma}{\sigma_o} \cong \frac{(\sigma_{\Delta o} B)^2}{2} \qquad (7b)$$

with $\sigma_{\Delta o}$ given by (7a).

Numbers:

Using the parameters supplied in the original problem:

L = 49 km
D_p = 0.5 psec/sqrt(km) = .0005 nsec/sqrt(km)
$P_\sigma = 10^{-12}$
B = 10.3125 GBd

We calculate, from (7a,b):

$\sigma_{\Delta o}$ = .00815 nsec = 8.15 psec

$\Delta\sigma/\sigma_o \cong .0035$

so the spreading corresponding to a probability of 10^{-12} is about 0.35 %.

<u>ISI Penalty due to PMD</u>:

We can estimate the PMD contribution to the ISI penalty by assuming that the PMD impulse response can be approximated by an Gaussian impulse response having the same rms width. With this assumption, we can utilize the treatment used in the spreadsheet model to calculate the ISI penalty due to a Gaussian impulse response:

$$P_{ISI}(\mathrm{dB}) = -10\,\mathbf{log_{10}}(2\,\mathrm{h_e} - 1) \quad \text{where}$$

$$\mathrm{h_e} = \mathrm{erf}\left(\frac{1}{\sqrt{8}\,\mathrm{B}\,\sigma_\Delta}\right) \qquad (8a)$$

where erf is the error function. Using the numbers above, we get:

PISI (dB) = 2.34×10^{-8}

ie: the PMD contribution to the ISI penalty is negligible for this particular case.