Notes on *a posteriori* probability (APP) metrics for LDPC

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Outline

Section 55.3.11.3 of draft D1.1 discusses decoding of LDPC code groups. This presentation outlines a method for calculation of exact LDPC metrics, with examples.

- General definition of APP metrics for iterative decoders.
- A simple example: PAM-2.
- APP metric for 1-D lattices.
- APP metric for 2-D and N-D lattices.
- Conclusions.

APP calculation for iterative decoders

Iterative decoders for LDPC and turbo-codes require *a posteriori* probabilities (APP's) as metrics, rather than direct channel observations.

- Usually expressed as a *log* ratio, and often referred to as LLR (log likelihood ratio).
- Let $\{b_0, b_1, b_2, ..., b_n\}$ be the set of bits mapped to a symbol a_k in constellation $\mathcal{A}$. We will denote the APP for, say bit b_0, as L_{b_0}.
- Consider the AWGN channel model: $\quad y = a_k + \upsilon \quad$, where a_k is the transmitted symbol, y is the channel observation, and υ is AWGN.
- We can write the definition of L_{b_0} as

$$
\begin{aligned}
L_{b_0} &\triangleq \log_e \left[\frac{p(b_0 = 0/y)}{p(b_0 = 1/y)} \right], \quad \text{and applying Bayes' rule, we get} \\
&= \log_e \left[\frac{p(y/b_0 = 0)p(b_0 = 0)}{p(y)} \frac{p(y)}{p(y/b_0 = 1)p(b_0 = 1)} \right], \\
&= \log_e \left[\frac{p(y/b_0 = 0)}{p(y/b_0 = 1)} \right].
\end{aligned}
$$

APP computation for PAM-2 in AWGN

- Consider the case of PAM-2 which takes $a_0 = +1$ when $b_0 = 0$, $a_1 = -1$ when $b_0 = 1$.
- We will make use of the conditional Gaussian pdf of a received symbol y in AWGN:

$$p(y/a) = \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{-\frac{(y-a)^2}{2\sigma^2}} \; ,$$

  where σ^2 is the noise variance.

- We can now derive the APP value for b_0 as

$$\begin{aligned} L_{b_0} &= \log_e \frac{p(y/b_0 = 0)}{p(y/b_0 = 1)} \; , \text{ as derived earlier,} \\ &= \log_e \frac{e^{-(y-a_0)^2/2\sigma^2}}{e^{-(y-a_1)^2/2\sigma^2}} \; , \\ &= \log_e \frac{e^{-(y-1)^2/2\sigma^2}}{e^{-(y+1)^2/2\sigma^2}} \; , \\ L_{b_0} &= \frac{2}{\sigma^2} \, y \; . \end{aligned}$$

- The APP value is a simpe scaled version of channel observation y. Notice that scaling is inversely proportional to noise variance.

APP for 1-D lattice: PAM-8 constellation [1]

To compute the APP's for PAM-8, notice that

$b_2b_1b_0$	PAM level
...	...
100	+9
010	+7
011	+5
001	+3
000	+1
110	-1
111	-3
101	-5
100	-7
010	-9
...	...

$$p(b_0 = 1/y) = \sum_k p(a_k/y), \qquad \text{where } a_k \in \mathcal{A}, \text{ s.t. } b_0 = 1$$

$$p(b_0 = 0/y) = \sum_j p(a_j/y), \qquad \text{where } a_j \in \mathcal{A}, \text{ s.t. } b_0 = 0$$

Using the above, we can now define the LLR for b_0 as

$$L_{b_0} \triangleq \log_e \left[\frac{p(b_0 = 0/y)}{p(b_0 = 1/y)}\right],$$

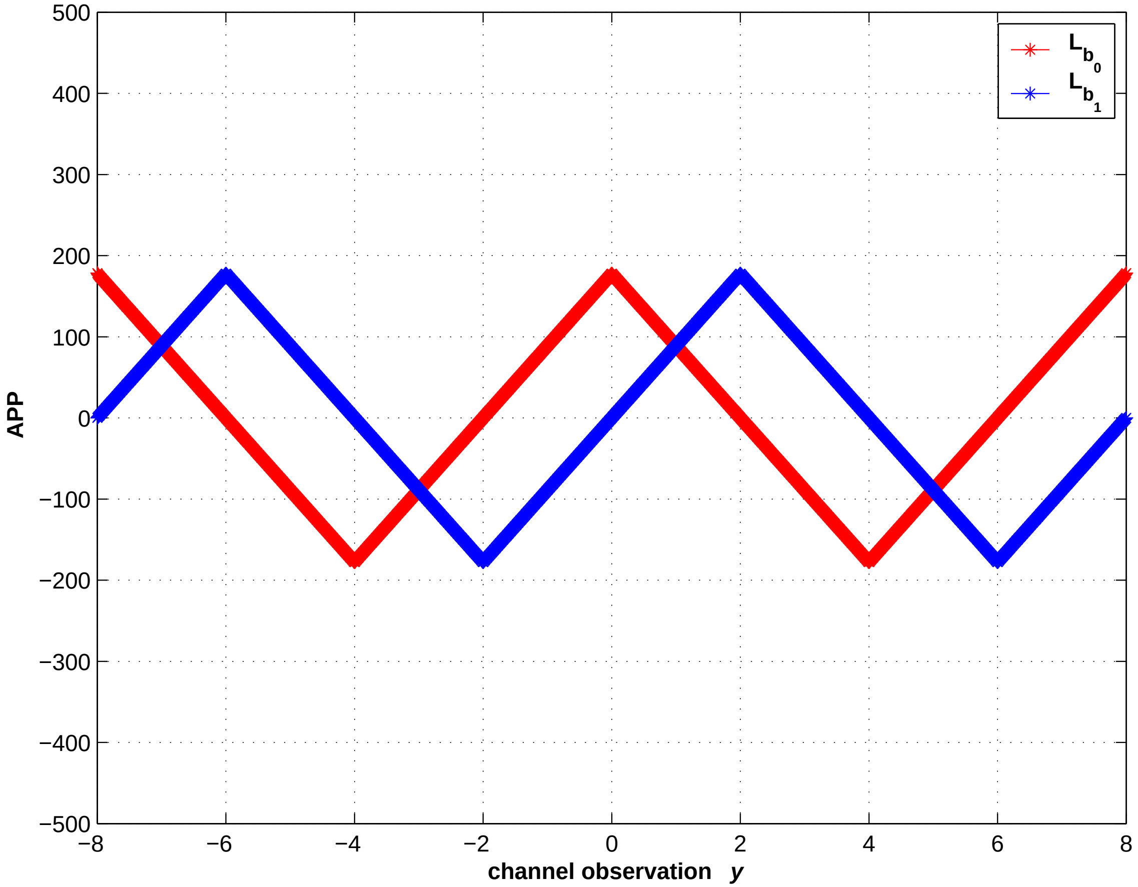
$$= \log_e \left[\frac{\sum_{a_k \in \mathcal{A},\ b_0=0} p(a_k/y)}{\sum_{a_j \in \mathcal{A},\ b_0=1} p(a_j/y)}\right].$$

Now, applying Bayes rule we get,

$$L_{b_0} = \log_e \left[\frac{\sum_{a_k \in \{\cdots,+7,+1,-1,-7,\cdots\}} e^{-\frac{(y-a_k)^2}{2\sigma^2}}}{\sum_{a_j \in \{\cdots,+5,+3,-3,-5,\cdots\}} e^{-\frac{(y-a_j)^2}{2\sigma^2}}}\right].$$

[1] S. Rao, R. Hormis, and E. Krouk, "The 4-D PAM-8 proposal for 10G-Base-T", http://www.ieee802.org/3/10GBT/public/nov03/rao_1_1103.pdf, Nov. 2003

APP for 1-D lattice: PAM-8 constellation



PAM-8: APP for bits b_0 and b_1, noise $\sigma = 0.15$

APP calculation for 2-D and N-D lattices

- In this case, the received symbol $\boldsymbol{y}$ and the constellation symbols, $\boldsymbol{a}_k \in \mathcal{A}$, can be viewed as vectors. Let us denote the components of vector $\boldsymbol{a}_k$ as $[a_{k0}\ a_{k1}\ \cdots]$. Similarly, $\boldsymbol{y} = [y_0\ y_1\ \cdots]$.

- We can write the APP for bit b_0 as

$$
\begin{aligned}
L_{b_0} &= \log_e \left[\frac{\sum_{\boldsymbol{a}_k \in \mathcal{A}, b_0=0} p(\boldsymbol{a}_k/\boldsymbol{y})}{\sum_{\boldsymbol{a}_j \in \mathcal{A}, b_0=1} p(\boldsymbol{a}_j/\boldsymbol{y})} \right], \\
&= \log_e \left[\frac{\sum_{\boldsymbol{a}_k \in \mathcal{A},\ b_0=0} p(\boldsymbol{y}/\boldsymbol{a}_k)}{\sum_{\boldsymbol{a}_j \in \mathcal{A},\ b_0=1} p(\boldsymbol{y}/\boldsymbol{a}_j)} \right], \quad \text{assuming that } p(\boldsymbol{a}_n) \text{ constant } \forall n.
\end{aligned}
$$

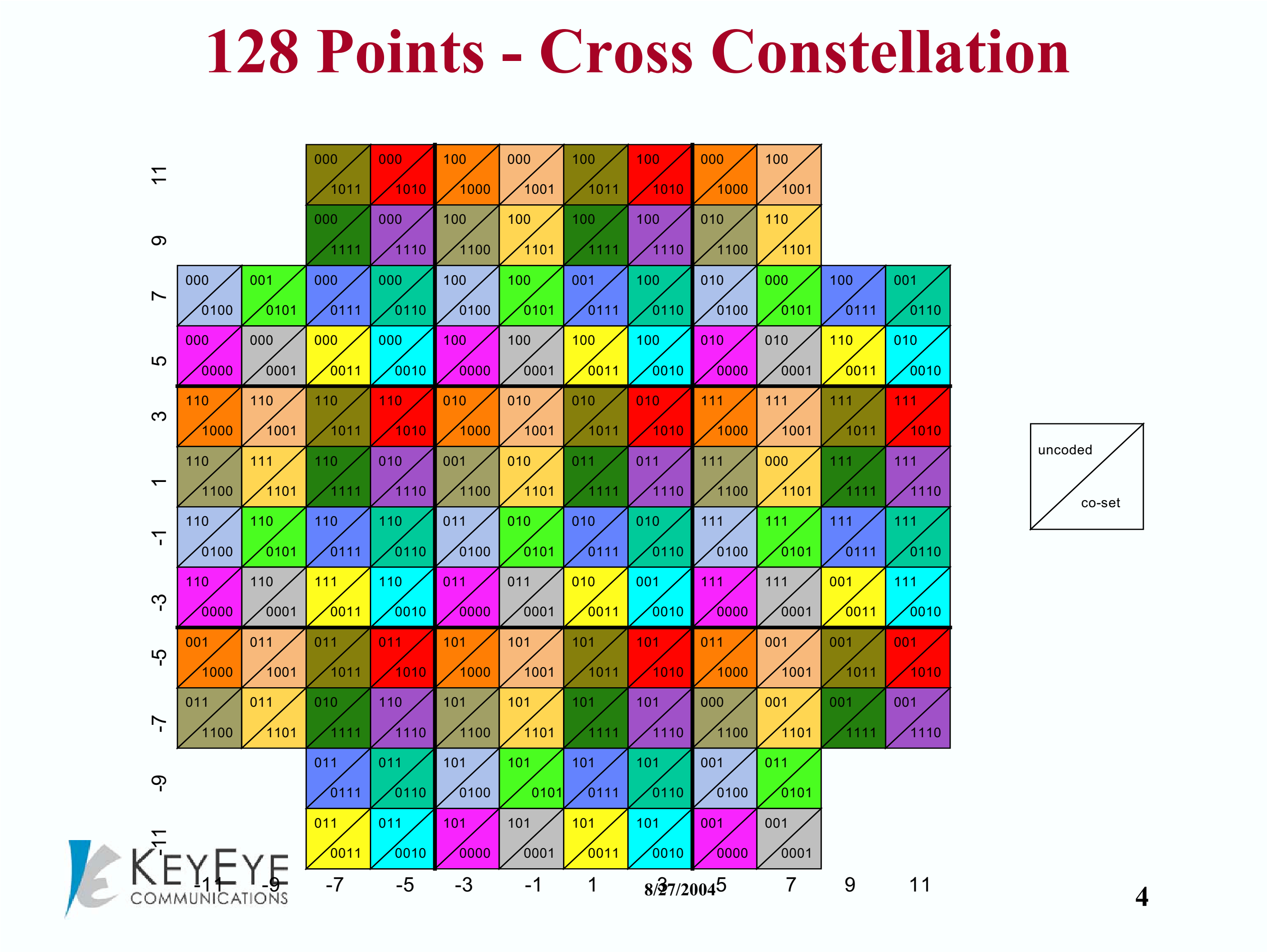
- Notice that $p(\boldsymbol{y}/\boldsymbol{a}_k) = p(y_0/a_{k0})\ .\ p(y_1/a_{k1})$, since $y_0, y_1, \cdots$ are conditionally independent given $a_{k0}, a_{k1}, \cdots$ respectively.

- We can then simplify the APP for b_0 as

$$
L_{b_0} = \log_e \left[\frac{\sum_{\boldsymbol{a}_k \in \mathcal{A},\ b_0=0} p(y_0/a_{k0})\ p(y_1/a_{k1})\ \cdots}{\sum_{\boldsymbol{a}_j \in \mathcal{A},\ b_0=1} p(y_0/a_{j0})\ p(y_1/a_{j1})\ \cdots} \right],
$$

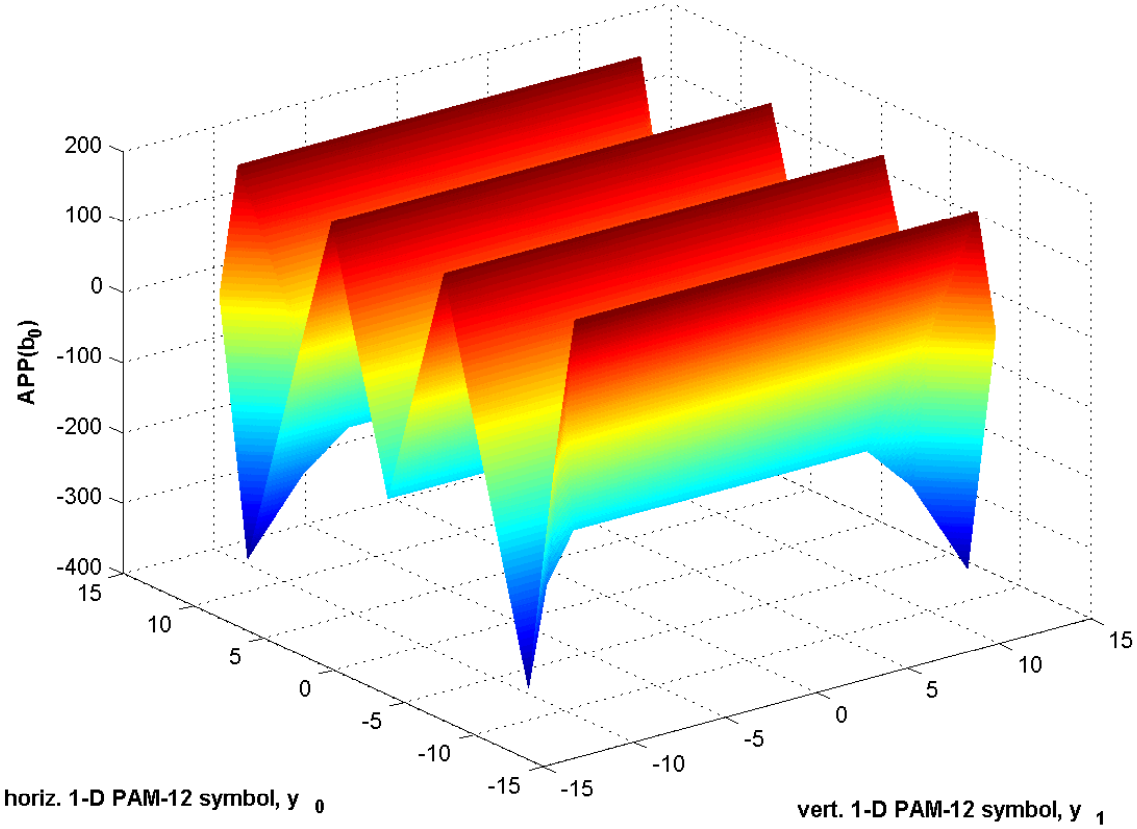
which involves products of 1-D conditional Gaussian pdf's .

APP for 2D-lattice: 2D-PAM12 Key-eye proposal [2]



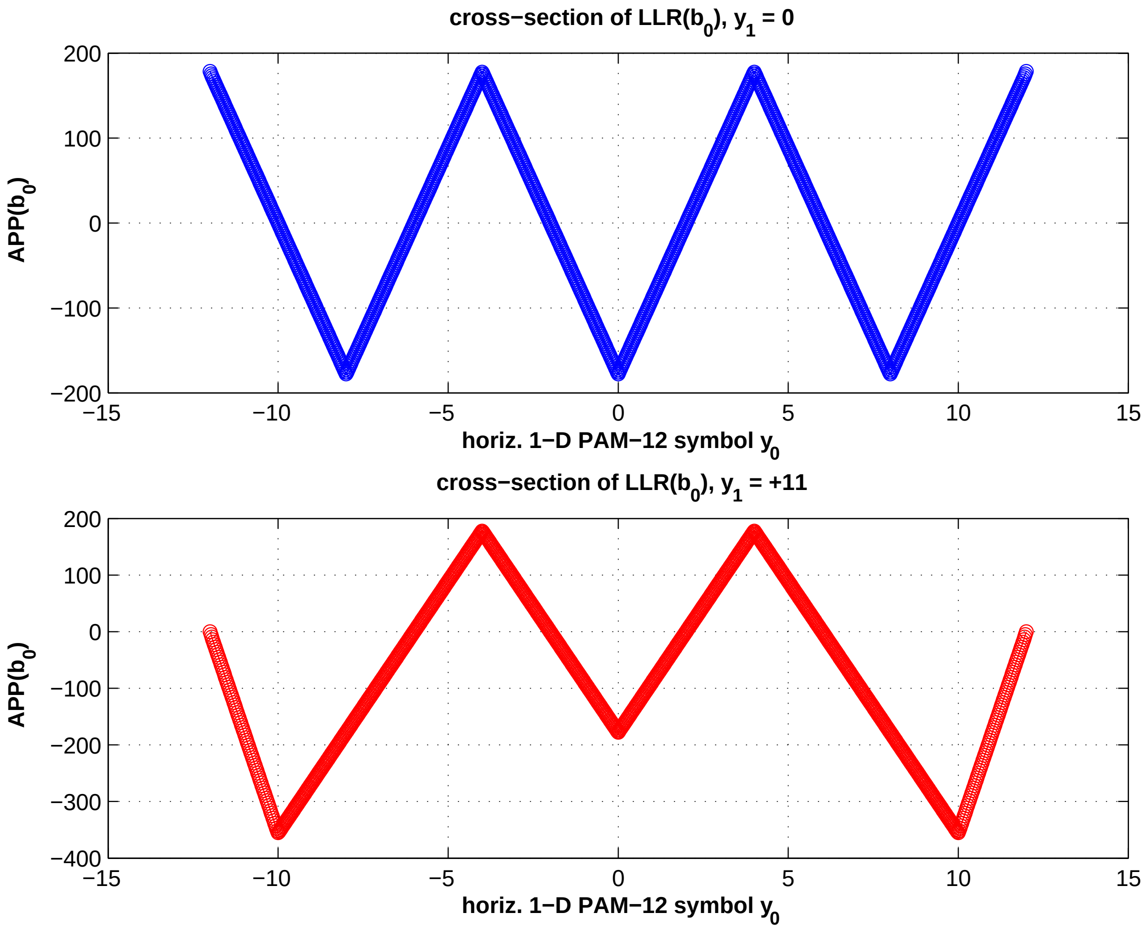
[2] Weizhuang Xin, “Modified 128 point cross constellation mapping”, http://www.ieee802.org/3/10GBT/email/msg01026.html , Aug. 2004

APP for 2D-lattice: 2D-PAM12 Key-eye proposal



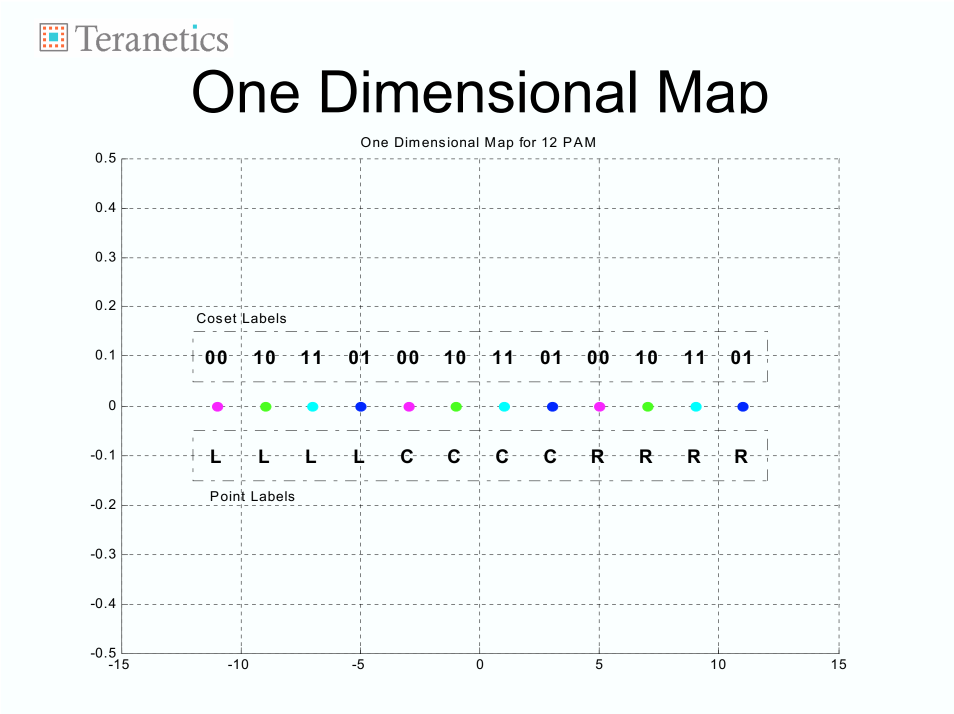
2-D APP function of b_0, noise $\sigma = 0.15$.

APP for 2D-lattice: 2D-PAM12 Key-eye proposal



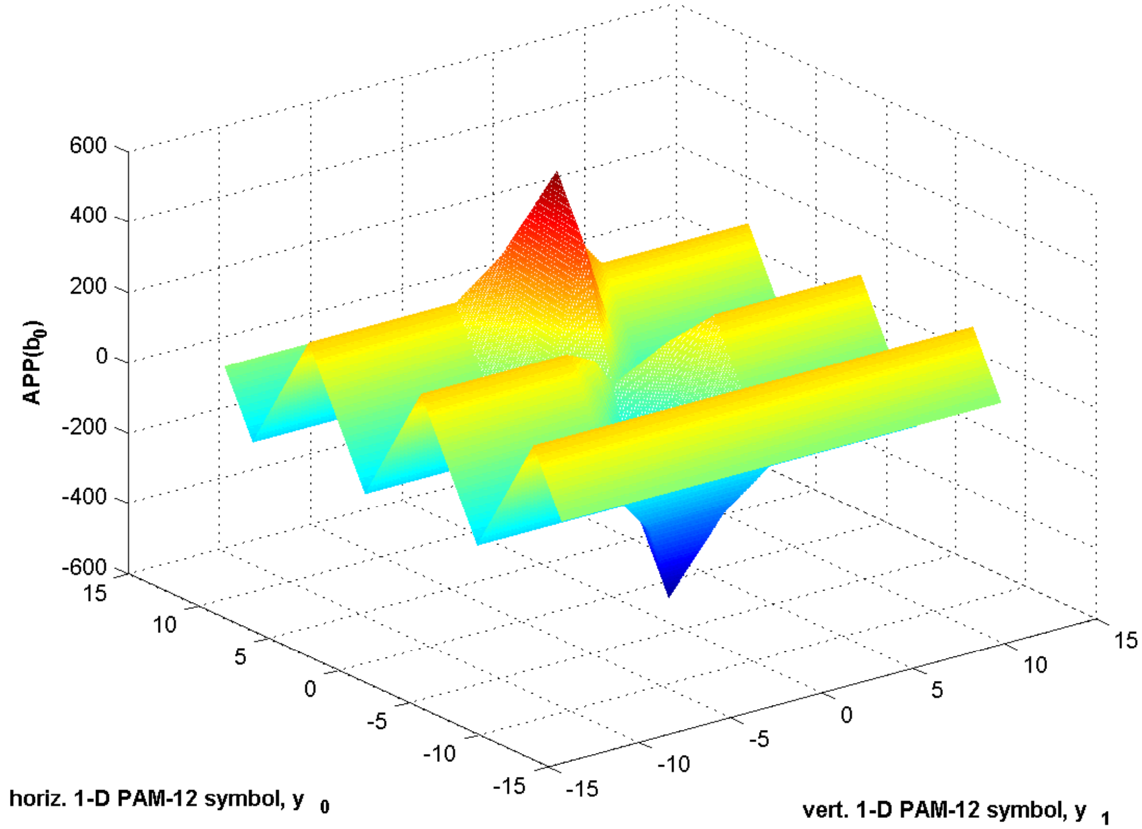
Cross-sections of 2-D APP function of b_0, at $y_0 = +11$ and $y_0 = 0$, noise $\sigma = 0.15$.

APP for 2D-lattices: 2D-PAM12 Teranetics proposal [3]



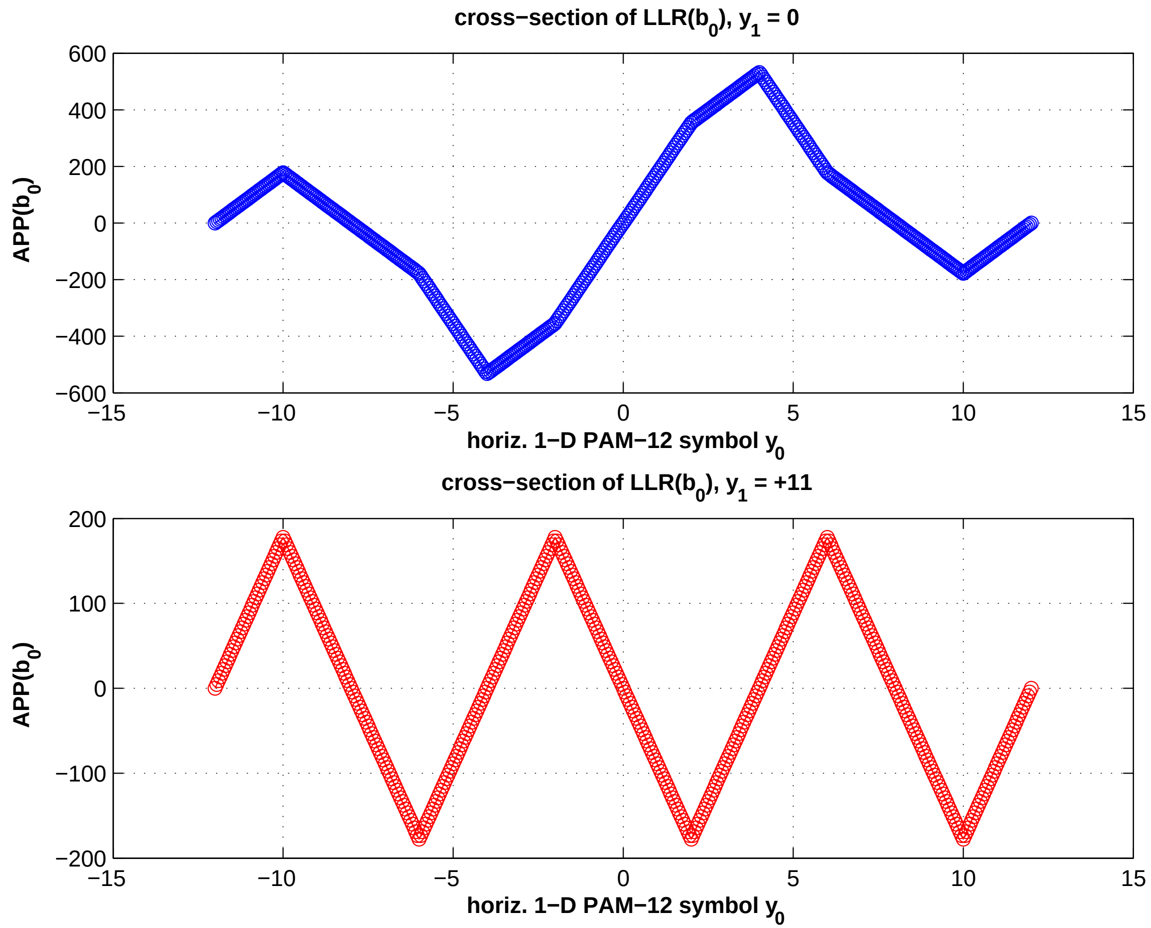
[3] D. Dabiri and J. Tellado, "Modifications to LDPC proposal offering lower symbol rate and lower latency", http://www.ieee802.org/3/an/public/mar04/dabiri_1_0304.pdf , Mar. 2004

APP for 2D-lattices: 2D-PAM12 Teranetics proposal



2-D APP function for bit b_0, noise $\sigma = 0.15$.

APP for 2D-lattices: 2D-PAM12 Teranetics proposal



Cross-section of APP function of b_0, $y_0 = +11$ and $y_0 = 0$, noise $\sigma = 0.15$.

Conclusions

- From first principles, we derived the generalized APP metric for multi-dimensional constellations.
- Non-rectangular constellations lead to APP's that are not always separable into 1-D functions …the 1-D functions would be approximations.

Recommendations:

- Use of 2-D (N-D) APP metrics, depending on the dimensionality of the constellation.
- The APP metrics tend to be approximately piece-wise linear (planar) – perhaps easy to map using LUT's or digital gates.
- The piece-wise linear APP metric surface should be documented.