Notes on *a posteriori* probability (APP) metrics for LDPC

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Raju Hormis, Xiaodong Wang
Columbia University, NY

e-mail: raju@ee.columbia.edu
Section 55.3.11.3 of draft D1.1 discusses decoding of LDPC code groups. This presentation outlines a method for calculation of exact LDPC metrics, with examples.

- General definition of APP metrics for iterative decoders.
- A simple example: PAM-2.
- APP metric for 1-D lattices.
- APP metric for 2-D and N-D lattices.
- Conclusions.
APP calculation for iterative decoders

Iterative decoders for LDPC and turbo-codes require \textit{a posteriori} probabilities (APP’s) as metrics, rather than direct channel observations.

- Usually expressed as a \textit{log} ratio, and often referred to as LLR (log likelihood ratio).
- Let \( \{b_0, b_1, b_2, ..., b_n\} \) be the set of bits mapped to a symbol \( a_k \) in constellation \( \mathcal{A} \). We will denote the APP for, say bit \( b_0 \), as \( L_{b_0} \).
- Consider the AWGN channel model: \( y = a_k + \nu \), where \( a_k \) is the transmitted symbol, \( y \) is the channel observation, and \( \nu \) is AWGN.
- We can write the definition of \( L_{b_0} \) as

\[
L_{b_0} \triangleq \log_e \left[ \frac{p(b_0 = 0|y)}{p(b_0 = 1|y)} \right], \quad \text{and applying Bayes’ rule, we get}
\]

\[
= \log_e \left[ \frac{p(y/b_0 = 0)p(b_0 = 0)}{p(y)} \frac{p(y)}{p(y/b_0 = 1)p(b_0 = 1)} \right],
\]

\[
= \log_e \left[ \frac{p(y/b_0 = 0)}{p(y/b_0 = 1)} \right].
\]
APP computation for PAM-2 in AWGN

• Consider the case of PAM-2 which takes $a_0 = +1$ when $b_0 = 0$, $a_1 = -1$ when $b_0 = 1$.

• We will make use of the conditional Gaussian pdf of a received symbol $y$ in AWGN:

\[ p(y/a) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-a)^2}{2\sigma^2}}, \]

where $\sigma^2$ is the noise variance.

• We can now derive the APP value for $b_0$ as

\[ L_{b_0} = \log_e \frac{p(y/b_0 = 0)}{p(y/b_0 = 1)}, \]

as derived earlier,

\[ = \log_e \frac{e^{-(y-a_0)^2/2\sigma^2}}{e^{-(y-a_1)^2/2\sigma^2}}, \]

\[ = \log_e \frac{e^{-(y-1)^2/2\sigma^2}}{e^{-(y+1)^2/2\sigma^2}}, \]

\[ L_{b_0} = \frac{2}{\sigma^2} y. \]

• The APP value is a simple scaled version of channel observation $y$. Notice that scaling is inversely proportional to noise variance.
To compute the APP’s for PAM-8, notice that

\[ p(b_0 = 1/y) = \sum_k p(a_k/y), \quad \text{where } a_k \in A, \text{ s.t. } b_0 = 1 \]

\[ p(b_0 = 0/y) = \sum_j p(a_j/y), \quad \text{where } a_j \in A, \text{ s.t. } b_0 = 0 \]

Using the above, we can now define the LLR for \( b_0 \) as

\[ L_{b_0} \triangleq \log_e \left[ \frac{p(b_0 = 0/y)}{p(b_0 = 1/y)} \right], \]

\[ = \log_e \left[ \frac{\sum_{a_k \in A, \ b_0 = 0} p(a_k/y)}{\sum_{a_j \in A, \ b_0 = 1} p(a_j/y)} \right]. \]

Now, applying Bayes rule we get,

\[ L_{b_0} = \log_e \left[ \frac{\sum_{a_k \in \{\ldots, +7, +1, -1, -7, \ldots\}} e^{-(y-a_k)^2/2\sigma^2}}{\sum_{a_j \in \{\ldots, +5, +3, -3, -5, \ldots\}} e^{-(y-a_j)^2/2\sigma^2}} \right]. \]

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APP for 1-D lattice: PAM-8 constellation

PAM-8: APP for bits $b_0$ and $b_1$, noise $\sigma = 0.15$
APP calculation for 2-D and N-D lattices

- In this case, the received symbol $y$ and the constellation symbols, $a_k \in \mathcal{A}$, can be viewed as vectors. Let us denote the components of vector $a_k$ as $[a_{k0} ~ a_{k1} ~ \cdots]$. Similarly, $y = [y_0 ~ y_1 ~ \cdots]$.

- We can write the APP for bit $b_0$ as

$$L_{b_0} = \log_e \left[ \frac{\sum_{a_k \in \mathcal{A}, b_0 = 0} p(a_k / y)}{\sum_{a_j \in \mathcal{A}, b_0 = 1} p(a_j / y)} \right] ,$$

which involves products of 1-D conditional Gaussian pdf’s.

- Notice that $p(y / a_k) = p(y_0 / a_{k0}) \cdot p(y_1 / a_{k1})$, since $y_0, y_1, \cdots$ are conditionally independent given $a_{k0}, a_{k1}, \cdots$ respectively.

- We can then simplify the APP for $b_0$ as

$$L_{b_0} = \log_e \left[ \frac{\sum_{a_k \in \mathcal{A}, b_0 = 0} p(y_0 / a_{k0}) \cdot p(y_1 / a_{k1}) \cdots}{\sum_{a_j \in \mathcal{A}, b_0 = 1} p(y_0 / a_{j0}) \cdot p(y_1 / a_{j1}) \cdots} \right] ,$$

assuming that $p(a_n)$ constant $\forall n$. 

Notice that $p(y / a_k) = p(y_0 / a_{k0}) \cdot p(y_1 / a_{k1})$, since $y_0, y_1, \cdots$ are conditionally independent given $a_{k0}, a_{k1}, \cdots$ respectively.
APP for 2D-lattice: 2D-PAM12 Key-eye proposal

128 Points - Cross Constellation

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APP for 2D-lattice: 2D-PAM12 Key-eye proposal

2-D APP function of $b_0$, noise $\sigma = 0.15$. 
Cross-sections of 2-D APP function of $b_0$, at $y_0 = +11$ and $y_0 = 0$, noise $\sigma = 0.15$. 
APP for 2D-lattices: 2D-PAM12 Teranetics proposal

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The Mapping of the Uncoded Bits in Two Dimensions

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3D. Dabiri and J. Tellado, “Modifications to LDPC proposal offering lower symbol rate and lower latency”,
2-D APP function for bit $b_0$, noise $\sigma = 0.15$. 
Cross-section of APP function of $b_0$, $y_0 = +11$ and $y_0 = 0$, noise $\sigma = 0.15$. 
Conclusions

• From first principles, we derived the generalized APP metric for multi-dimensional constellations.

• Non-rectangular constellations lead to APP’s that are not always separable into 1-D functions ... the 1-D functions would be approximations.

Recommendations:

• Use of 2-D (N-D) APP metrics, depending on the dimensionality of the constellation.

• The APP metrics tend to be approximately piece-wise linear (planar) – perhaps easy to map using LUT’s or digital gates.

• The piece-wise linear APP metric surface should be documented.