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Re:	IEEE P802.16e/D4-2004	
Abstract	Proposing a new DL preamble sequence design for use in IEEE P802.16e/D4-2004	
Purpose	To improve the cell search and cell synchronization in mobile environment and reduce computational complexity for extended battery standby time.	
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# Preamble Sequence For Fast Cell Search, Low Computational Complexity, And Low PAPR

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## 1. Introduction

The DL preambles currently defined in 802.16e require that the MSSs capture the preamble symbols and correlate with 100+ PN sequences in the frequency domain to determine *IDcell* and Segment of the specific sector. In this contribution we propose a new DL preamble design for 1024-FFT, 512-FFT, and 128-FFT OFDMA PHY that aims at providing a structural generation of preamble sequence to facilitate fast cell searching and reduced computational complexity. The new DL preamble design is based on the Chu and Frank-Zadoff CAZAC sequences defined in 802.16SCa and are used in the frequency domain. The *IDcell* parameter is associated with the code phase of the CAZAC sequence in the frequency domain and the Segment ID is detected using hypothesis testing of FCH and common segment signaling.

Benefits of using the proposed DL preamble design are:

- Low peak-to-average power ratio due to inherent CAZAC properties,
- Provide for fine symbol timing computation due to orthogonality of the CAZAC sequence in the estimation of CIR (channel impulse response),
- Fast cell searching due to the use of a single known sequence both in frequency and time domain. Sequence matching can be done solely in frequency domain and computations are multiplierless (CORDICs and adders only).
- Integral neighboring cell/sector scanning during preamble detection because cells and sectors are differentiated using CAZAC code-phases, which are assigned to the *IDcell* parameter.
- Allows for accurate MSS fine time tracking and delay-time adjustment in mobile environment via CIR arrival time estimation. It also allows for accurate control of UL TX slot timing during MSS HO to a neighboring BS because a single scan of the captured preamble symbol produces neighbor BS CIR, time of arrival, RSSI, etc.
- Lowest computational complexity during sleep-mode scanning for neighboring cells. It provides a mechanism for extended battery stand-by time.

## 2. Proposed Solutions

Some properties of CAZAC sequences are described and derived in the Appendix. The most prominent one is that “constant-amplitude zero-autocorrelation” property is preserved both in time and frequency domains. In this contribution we propose that the preamble be split into four preamble carrier-sets in PUSC configuration.

In IEEE802.16a SCa 8.3.1.3.2.2 two sequences are used. Chu sequences are defined as

$$x(n) = \exp(j2\pi n^2 / L) \tag{1}$$

where

$$L = 8, 32, 128, 512, \dots \tag{2}$$

where  $L$  is the length of the sequence,  $L=8, 32, 128, 512, \dots$

The Frank-Zadoff sequences are also defined in (1) but the phase is defined as

$$x(n) = \exp(j2\pi n^2 / L) \sqrt{\frac{L}{m}} \exp(j\pi n^2 / m) \tag{3}$$

where  $L$  is the length of the sequence,  $L=16, 64, 256, 1024, \dots$

In this contribution we use Chu and Frank-Zadoff CAZAC sequences to form preamble sequences. For clarity, only 1024-FFT OFDMA is described here. In the case of PUSC configuration, there are four preamble carrier-sets. The subcarriers are modulated using a boosted PSK modulation with a CAZAC sequence cyclicly shifted with a code phase defined by  $ID_{cell}$ .

The preamble carrier-sets are defined using the following formula:

$$PreambleCarrierSet_m = m + 4 * k \tag{4}$$

where:

$PreambleCarrierSet_m$  specifies all subcarriers allocated to the specific preamble

$m$  is the number of the preamble carrier-set indexed 0..3

$k$  is a running index

where each segment is assigned one of the four possible preamble carrier-sets. The segment assignments are

- Segment 0 uses preamble carrier-set 0 and additionally modulate carrier-set 3 with a segment-specific code-phase equal to  $ID_{cell} 0$  as a reference signal.
- Segment 1 uses preamble carrier-set 1 and additionally modulate carrier-set 3 with a segment-specific code-phase equal to  $ID_{cell} 10$  as a reference signal.
- Segment 2 uses preamble carrier-set 2 and additionally modulate carrier-set 3 with a segment-specific code-phase equal to  $ID_{cell} 20$  as a reference signal.

Let the 1024-FFT OFMA sampling rate be 10MHz at Nyquist rate. The basic preamble time-domain symbol rate is 5MHz. The frequency-domain components are composed of a Chu sequence described in (1) and (2) of length 128 that is zero-inserted to length 512 by inserting CAZAC symbols one for every three frequency subcarriers. Note that due to guard bands and channel select filtering, we cannot use CAZAC sequences of length 1024 at Nyquist rate of 10MHz samples. Doing so will unavoidably violate transmit spectrum mask. In Appendix we established that a time-domain CAZAC sequence at symbol rate (5MHz) introduces a CAZAC sequence in frequency domain (after spectrum folding). Its frequency-domain CAZAC sequence can be computed using a 512-FFT operation instead of 1024.

To preserve time-domain CAZAC characteristics at 10MHz symbol rate, it will unavoidably introduce spectrum folding in the frequency domain. In the following section we propose a method to preserve CAZAC sequence characteristics of the folded frequency spectrum so that CAZAC is preserved in both frequency and time domains.

The proposed construction of the CAZAC sequence aims at reconstructing the 1024 subcarriers using the 3:1 zero-inserted 512-element frequency-domain CAZAC sequence of a 128-element Chu sequence so that after spectrum folding, the folded 512 spectral components form the frequency-domain CAZAC sequence of the Chu sequence.

Let  $\mathbf{c}_k$  denote the time-domain 512-element CAZAC sequence and its frequency-domain CAZAC sequence be denoted as  $\mathbf{C}_k$  (512 elements) and expressed as

$$\mathbf{C}_k = \text{fft}(\mathbf{c}_k) \tag{5}$$

where  $k$  denotes the preamble carrier-set.  $\mathbf{c}_k$  and  $\mathbf{C}_k$  form a time-frequency pair and their relationship is expressed as

$$\mathbf{c}_k = \text{ifft}(\mathbf{C}_k) \tag{6}$$

In IEEE P802.16e/D4 1024-FFT OFDMA has 86 guard subcarriers on the left-hand side and 87 on the right-hand side. The DC subcarrier resides on index 512. The construction procedures of assembling  $\mathbf{C}_k$  and  $\mathbf{c}_k$  of the left- and right-hand sides 1024-FFT OFDMA preambles are

$$\mathbf{C}_k = \text{fft}(\mathbf{c}_k) \tag{7}$$

$$\mathbf{c}_k = \text{ifft}(\mathbf{C}_k) \tag{8}$$

$$\mathbf{C}_k = \text{fft}(\mathbf{c}_k) \tag{9}$$

$$\mathbf{c}_k = \text{ifft}(\mathbf{C}_k) \tag{10}$$

The final reconstructed 1024-FFT frequency components of the preamble symbol are

$$\mathbf{C}_k = \text{fft}(\mathbf{c}_k) \tag{11}$$

and its final reconstructed 1024 time-domain preamble sequence at Nyquist rate is

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \tag{12}$$

Note that after spectrum folding due to subsampling at symbol rate in the time domain, the resulting folded frequency spectral components of even-numbered samples are

$$x_e(n) = \sum_{k=0}^{N/2-1} X(k) e^{j2\pi kn/N} + \sum_{k=N/2}^{N-1} X(k) e^{j2\pi kn/N} \tag{13}$$

and where the overlapped area has the following relationship (see Appendix),

$$x_e(n) = \sqrt{2} \sum_{k=0}^{N/4-1} X(k) e^{j2\pi kn/N} + \sqrt{2} \sum_{k=N/4}^{N/2-1} X(k) e^{j2\pi kn/N} \tag{14}$$

Note also that overlapped area of odd-numbered samples has the following relationship (see Appendix),

$$x_o(n) = \sqrt{2} \sum_{k=0}^{N/4-1} X(k) e^{j2\pi kn/N} + \sqrt{2} \sum_{k=N/4}^{N/2-1} X(k) e^{j2\pi kn/N} \tag{15}$$

Doing so, the reconstructed frequency sequence has mild distortion compared to the desired CAZAC sequence and the time sequence has the lowest PAPR whereby even- and odd-numbered samples conforms to CAZAC sequences and are mildly distorted due to guard bands. The nominal PAPR of the time-domain sequence is less than 3dB at all different code-phases. It is important to note that the frequency components of the reconstructed 1024-FFT in the preamble sequence are constant-amplitude to facilitate channel estimation.

The *IDcell* allocation are done via assigning CAZAC code phases of cyclic shift of the  $x(n)$  sequence and forming the time-domain sequence in the same manners described in (7)-(12). Figure 1 shows an example of the subcarrier allocations of the preamble sequence in segment 0. Figure 2 shows the corresponding time waveform.

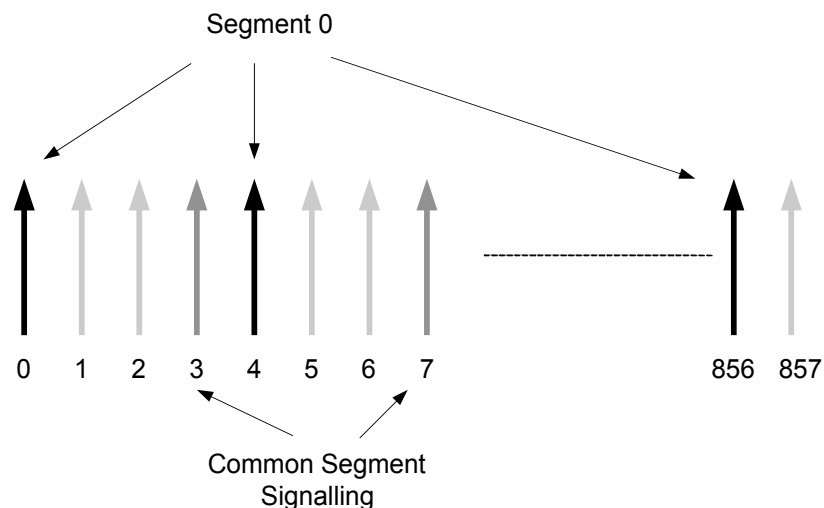


Figure 1. Example of segment 0 preamble subcarrier allocation.

Another important characteristic of Chu and Frank-Zadoff sequences is their duality behavior in both the time and frequency domain. Figure 2 and 3 show an example of matched filtering results of a Chu sequence in both time and frequency domain in an SUI-3 RF multipath environment. The estimated

CIR in time domain meets the resulting matched filtering outputs in the frequency domain. In other words, the detection of arrival time and CIR can be processed solely in the frequency domain. It is due to the properties of a Chu sequence that translation in time becomes translation in frequency, vice versa. Because of this, a MSS can accurately track the DL arrival time drift during preamble detection in a mobile environment and perform responsive timing adjustment so that UL performance does not degrade due to inaccurate UL transmit timing. Additionally, the matched filtering signal processing in frequency domain does not require IQ complex multipliers. Only CORDICs and adders are needed. It is because CAZAC sequence is a unit-amplitude complex sequence. The complexity can be further reduced after exploring symmetries of Chu and Frank-Zadoff sequences. For example, a 16-element Frank-Zadoff sequence is a sequence of  $\pm 1$  and  $\pm j$ . Matched filtering operations require and additions and subtraction only.

It is also important to point out that a single scan of CAZAC matched filtering in frequency domain of the captured preamble symbol yields all information regarding neighbor BS CIR, time of arrival, RSSI, frequency usage, etc. It is best suited for HO and sleep-mode scanning. Also the capability of acquiring time of arrival of neighboring BSs is critical in adjusting UL timing when in HO so that UL multi-user timing violation is minimized. It also helps MSS adjust timing easily during high-speed vehicular motion where the timing advance/retard can be adjusted from the detection of DL preamble time of arrival.

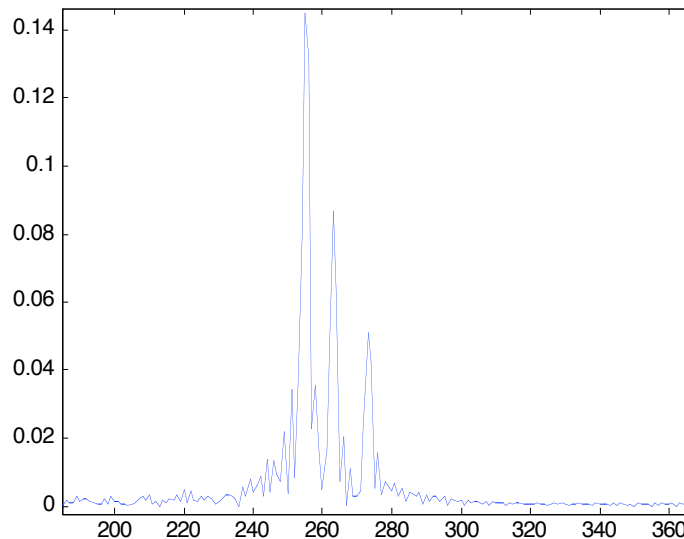


Figure 2. Chu sequence matched filtering in the time domain.

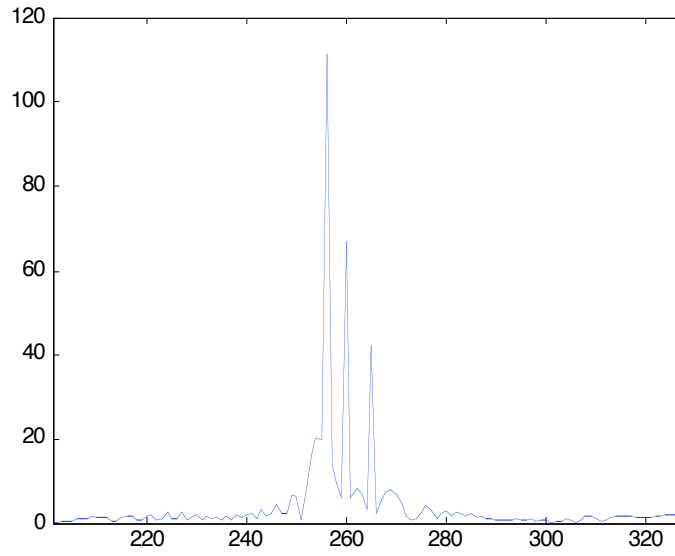


Figure 3. Chu sequence matched filtering in the frequency domain.

In addition to segment modulation, to robustly identifying code phases of the CAZAC sequences in different segments, carrier-set 3 is used as a common segment signaling. The common segment signaling is used as a timing and frequency reference signal for robust identification of code phase of *IDcell*. It is especially useful in large multi-path delay environment whereby MSS can examine the CAZAC matched results of carrier-set 3 and correlate with results of the three segments to robustly determine the *IDcell* parameters of all three segments. The carrier-set 3 is modulated without boosting so as not to degrade PAPR too much. The common segment signaling is for detection and not for channel estimation. Figure 4 and 5 show an example of an overlapping sector at the edge of the cell with SUI-3 multi-path delay.

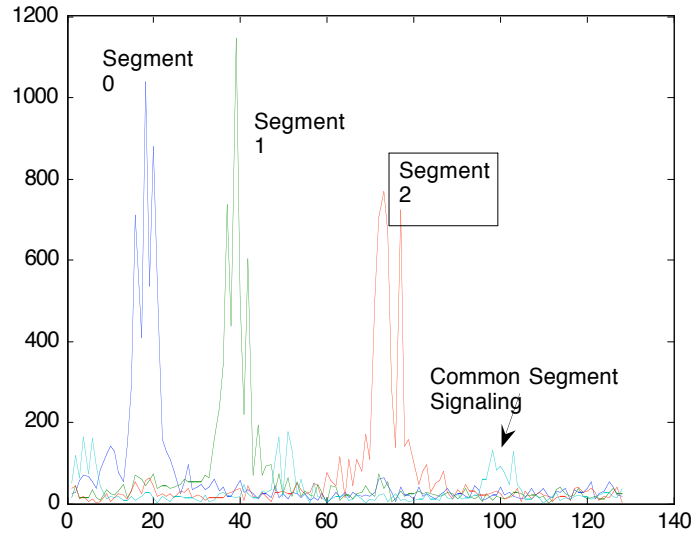


Figure 4. CAZAC matched filtering outputs of all four carrier-sets of an MSS on cell-edge with overlapping sectors from neighboring three BSs in SUI-3 environment.

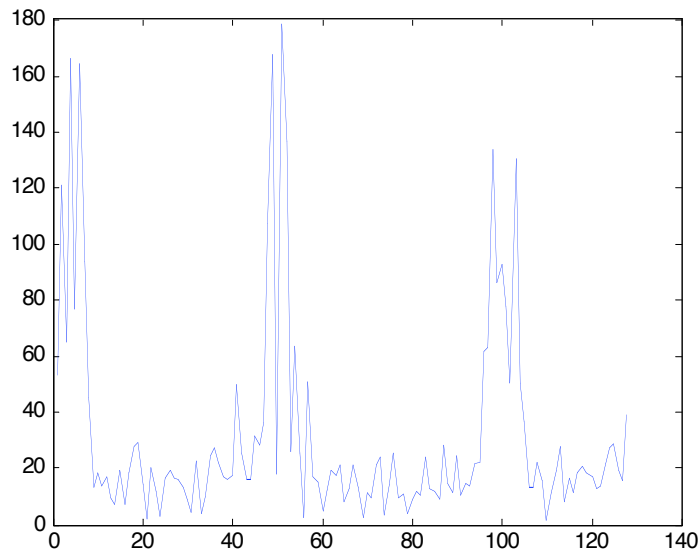


Figure 5. Common segment signaling of carrier-set 3 on cell-edge from overlapping sectors of neighboring three BSs with different IDcells.

The simulated PAPR in various modes are summarized in Table 1.



Table 1. Peak-to-Average Power Ratio (PAPR) in all configurations.

Configuration	Sequence Type	PAPR max (dB)
1024-FFT FUSC	512 Chu	2.6
1024-FFT PUSC	128 Chu	3.5
512-FFT FUSC	256 Frank-Zadoff	2.7
512-FFT PUSC	64 Frank-Zadoff	3.6
128-FFT FUSC	64 Frank-Zadoff	3.0
128-FFT PUSC	16 Frank-Zadoff	4.7

### 3. Performance Benefits

Use of a CAZAC sequence in the frequency domain guarantees constant-amplitude time waveform. However, due to the exclusion of guard bands, direct usage of a CAZAC sequence cannot be employed. The proposed DL preamble design inherits the CAZAC properties and introduces spectrum folding so that the time waveform maintains a very low-degree of amplitude fluctuation, with the source of fluctuation coming from the exclusion of guard bands and DC subcarrier.

Benefits of the preamble design when compared to 802.16d 8.4.6.1.1 are

1. Mild increase of PAPR (from 0dB) while maintaining constant-amplitude in the frequency domain to facilitate channel estimation.
2. Allows for fast cell searching because a single CAZAC sequence is used. *IDcell* and segment are identified via the code-phase of the CAZAC sequence and frequency offset identification via common segment signaling and FCH detection.
3. Immediate channel impulse response (CIR) identification due to the use of a CAZAC sequence. Note that because of the duality of CAZAC in both time- and frequency-domain, the CIR identification can be performed either in time- or frequency- domain.
4. Provide a mechanism for fine time-tracking in MSS and allow for built-in timing correction for MSS mobility control that minimize the need for periodic ranging requests. The duality of CAZAC sequence introduce phenomenon that translation in time becomes translation in frequency. In other words. Tracking of CIR movements in the DL preamble matched-filtering in the frequency domain provides exact information of timing delay from MSS movement.
5. Capable of performing neighbor BS scanning during normal preamble reception due to the fact that neighbor BSs use the same CAZAC sequence with different code phases.
6. Very low computational complexity. It allows for extended battery standby time.

## 4. Specific text changes

Replace IEEE P802.16e/D4 8.4.6.1.1 as follows

For the FFT sizes of 1024, 512, and 128, the preamble sequences are derived from Frank-Zadoff [xx] or Chu [xx] sequences and possess CAZAC (Constant Amplitude Zero Auto-correlation) properties.

Table 308. Preamble CAZAC sequences.

FFT size	Configuration	Sequence Type	Sequence length $L$
1024	FUSC	Chu	512
1024	PUSC	Chu	128
512	FUSC	Frank-Zadoff	256
512	PUSC	Frank-Zadoff	64
128	FUSC	Frank-Zadoff	64
128	PUSC	Frank-Zadoff	16

The Chu sequence generation is expressed as

$$C(n) = \exp\left(-j\pi \frac{a^2 n^2}{L}\right) \quad (x)$$

The Frank-Zadoff sequence generation is expressed as

$$Z(k) = \exp\left(-j\pi \frac{k^2}{L}\right) \exp\left(-j\pi \frac{k^2}{L}\right)$$

The preamble carrier-sets are defined using the following formula:

where

$PreambleCarrierSet_s$  specifies all subcarriers allocated to the specific preamble

$s$  is the number of preamble carrier-set indexed 0...3

$k$  is a running index 0...

Each segment uses a preamble composed of a carrier-set out of the 4 available carrier sets in the following manner. The DC carrier will not be modulated at all and the appropriate element will be discarded, therefore DC carrier shall always be zeroed and guard bands be zeroed.

- Segment 0 uses preamble carrier-set 0 and additionally modulates carrier-set 3 with a segment-specific code-phase equal to  $ID_{cell} 0$  as a reference signal.
- Segment 1 uses preamble carrier-set 1 and additionally modulates carrier-set 3 with a segment-specific code-phase equal to  $ID_{cell} 10$  as a reference signal.
- Segment 2 uses preamble carrier-set 2 and additionally modulates carrier-set 3 with a segment-specific code-phase equal to  $ID_{cell} 20$  as a reference signal.

Therefore each segment eventually modulates each 4'th subcarrier and collectively modulates carrier-set 3 without boosting for common segment signaling to establish a segment timing and frequency reference. Figure X depicts an example of the preamble of segment 0.

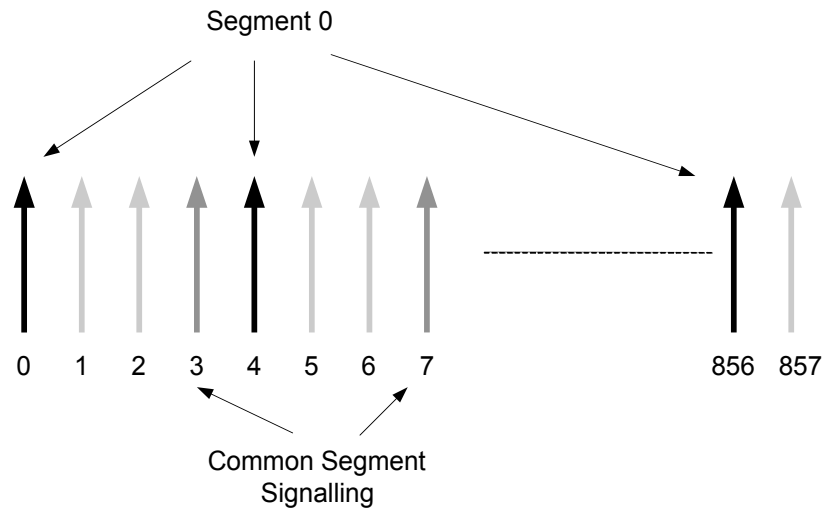


Figure X. Downlink PUSC basic structure

For FUSC, all subcarriers are modulated without segmentation. The preamble sequences for FUSC and PUSC are constructed as follows.

**8.4.6.1.1.1 1024-FFT OFDMA DL Preamble Sequence Generation**

In FUSC mode, the preamble modulation data of 1024 physical subcarriers are assembled in such a way that the folded frequency spectrum of the 2x subsampled time waveform closely resembles a 512-element Chu sequence while maintaining constant amplitude of all preamble elements for channel estimation. The assembling process uses a 512-element Chu sequence described in the last section and the procedures are

where

and  $ID_{cell}$  is between 0 and 31 as defined in X.X.X.X.  $G_{left}$  and  $G_{right}$  are the numbers of guard subcarriers on the left- and right-hand sides, respectively, as defined in Table 309b.  $x$  is a 512-element Chu sequence defined earlier in (x).

In PUSC mode, the preamble sequence is defined the same as in (x)-(x) but assembling process uses a 128-element Chu sequence instead and it is described as

where  $s$  denotes the preamble carrier-set and  $G_{left}$  and  $G_{right}$  are the numbers of guard subcarriers on the left- and right-hand sides, respectively, as defined in Table 308b. For common segment signaling using carrier-set 3, segment 0 uses a fixed code phase corresponding to  $ID_{cell}$  0. Segment 1 uses a fixed code phase corresponding to  $ID_{cell}$  10. Segment 2 uses a fixed code phase corresponding to  $ID_{cell}$  20. Preamble modulation is described in 8.4.9.4.3.1.

**8.4.6.1.1.2 512-FFT OFDMA DL Preamble Sequence Generation**

In FUSC mode, the preamble modulation values of 512 physical subcarriers are assembled in such a way that the folded frequency spectrum of the 2x subsampled time waveform closely resembles a 256-element Frank-Zadoff CAZAC sequence while maintaining constant amplitude of all preamble elements for channel estimation. The assembling process uses a 256-element Frank-Zadoff sequence described in the last section and the procedures are

where

and  $ID_{cell}$  is between 0 and 31 as is defined in X.X.X.X.  $G_{left}$  and  $G_{right}$  are the numbers of guard subcarriers on the left- and right-hand sides, respectively, as defined in Table 309c.  $x$  is a 256-element Frank-Zadoff sequence defined earlier in (x).

In PUSC mode, the preamble sequence is defined exactly the same as in (x)-(x) but assembling process uses a 64-element Frank-Zadoff sequence instead and is described as

where  $s$  denotes the preamble carrier-set and  $G_{left}$  and  $G_{right}$  are the numbers of guard subcarriers on the left- and right-hand sides, respectively, as defined in Table 308c. For common segment signaling using carrier-set 3, segment 0 uses a fixed code phase corresponding to  $ID_{cell}$  0. Segment 1 uses a fixed code

phase corresponding to *IDcell* 10. Segment 2 uses a fixed code phase corresponding to *IDcell* 20. Preamble modulation is described in 8.4.9.4.3.1.

**8.4.6.1.1.3 128-FFT OFDMA DL Preamble Sequence Generation**

In FUSC mode, the preamble modulation values of 128 physical subcarriers are assembled in such a way that the folded frequency spectrum of the 2x subsampled time waveform closely resembles a 64-element Frank-Zadoff CAZAC sequence while maintaining constant amplitude of all preamble elements for channel estimation. The assembling process uses a 64-element Frank-Zadoff sequence described in the last section and the procedures are

where

and *IDcell* is between 0 and 31 as is defined in X.X.X.X.  $g_{left}$  and  $g_{right}$  are the numbers of guard subcarriers on the left- and right-hand sides, respectively, as defined in Table 309d.  $x_{64}$  is a 64-element Frank-Zadoff sequence defined earlier in (x).

In PUSC mode, the preamble sequence is defined exactly the same as in (x)-(x) but assembling process uses two 16-element Frank-Zadoff sequence instead and are described as

where *s* denotes the preamble carrier-set and  $x_{16}$  is the sequence generated by the 16-point FFT of the 16-element Frank-Zadoff sequence with amplitude normalized to 1.  $g_{left}$  and  $g_{right}$  are the numbers of guard subcarriers on the left- and right-hand sides, respectively, as defined in Table 308d. For common segment signaling using carrier-set 3, segment 0 uses a fixed code phase corresponding to *IDcell* 0. Segment 1 uses a fixed code phase corresponding to *IDcell* 10. Segment 2 uses a fixed code phase corresponding to *IDcell* 20. Preamble modulation is described in 8.4.9.4.3.1.

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End of text change

Add the following test to 8.4.9.4.3

Begging text addition

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For 1024-, 512-, and 128-FFT modes, the pilots are modulated without PN spreading. The pilot subcarriers shall be modulated according to the following formula:

End of text change.

Add the following text to 8.4.9.4.3.1

Beginning text addition

For 1024-, 512-, and 128-FFT modes, the preamble shall be modulated according to the following formula:

$$\begin{aligned}
 & \dots \sqrt{\dots} \\
 & \dots \sqrt{\dots}
 \end{aligned}$$

For common segment signaling using carrier-set 3, the preamble shall be modulated according to the following formula:

$$\begin{aligned}
 & \dots \\
 & \dots \\
 & \dots
 \end{aligned}$$

End of text change.

## Appendix: Mathematical Background

Some properties of CAZAC [1] (constant-amplitude zero autocorrelation) sequences are presented in this section.

Let  $\{c_k, k=0..L-1\}$  be a CAZAC sequence and define the cyclic shift operator matrix  $\mathbf{M}$  as

$$\mathbf{M} = \begin{bmatrix} c_0 & c_1 & \dots & c_{L-1} \\ c_1 & c_2 & \dots & c_0 \\ \dots & \dots & \dots & \dots \\ c_{L-1} & c_0 & \dots & c_{L-2} \end{bmatrix} \tag{16}$$

where  $\{e_k, k=0..L-1\}$  are the standard basis vectors of the  $L$ -dimensional complex space  $\mathbb{C}^L$ . Define the circulant matrix  $\mathbf{C}$  of the CAZAC sequence as

$$\mathbf{C} = \mathbf{F} \mathbf{C} \mathbf{F}^H \tag{17}$$

where  $\mathbf{c}$  is a column vector formed by the CAZAC sequence

Define the Fourier matrix as

$$F_{lm} = \frac{1}{\sqrt{L}} e^{-j2\pi l m / L} \tag{18}$$

where  $L$  is the rank of the matrix. It can be shown that a circulant matrix can be uniquely expressed as [2]

$$\mathbf{C} = \mathbf{F} \mathbf{\Lambda} \mathbf{F}^H \tag{19}$$

where  $\mathbf{\Lambda} = \text{diag}\{g_0, g_1, \dots, g_{L-1}\}$  are the eigenmatrix of the circulant matrix.

A zero-autocorrelation sequence is characterized by its identity autocorrelation matrix, or

$$R_{ZZ} = \mathbf{I} \tag{20}$$

From (20) we can derive

$$\mathbf{C} \mathbf{e}_0 = \mathbf{e}_0 \tag{21}$$

In other words, eigenvalues of a circulant matrix form by a ZAC (zero-autocorrelation) sequence have equal amplitudes, or  $|g_l| = 1$ . Furthermore, these eigenvalues constitute the frequency spectral components of the ZAC sequence as is evident in the following equation,

$$\mathbf{C} \mathbf{e}_l = g_l \mathbf{e}_l \tag{22}$$

where  $\mathbf{e}_0$  is the 0<sup>th</sup> standard basis vector of the complex vector space and  $\mathbf{e}_l$  is the column vector formed by the eigenvalues of  $\mathbf{C}$ .

**Claim 1: If  $c$  is a CAZAC sequence, then its frequency domain spectral components also form a CAZAC sequence [6] (necessary condition).**

Proof:

Let  $W$  be the eigenmatrix of the cyclic shift operator matrix  $M$  defined in (16). It can be proved that  $W$  is a unitary matrix. Because  $M$  is a real matrix, we have

$$W^H = W^{-1} \tag{23}$$

Observe that for  $k=0, \dots, L-1$ ,

$$W_{k,l} = W_{k,l-L} \tag{24}$$

It has been shown earlier that the eigenvalues of the circulant matrix  $C$  of a CAZAC sequence has equal amplitude. With (24) it is proven that the sequence is a CAZAC sequence.

Q.E.D.

**Claim 2: If  $g$  is a CAZAC sequence in the frequency domain, then its corresponding time-domain sequence is also a CAZAC sequence [6] (sufficient condition).**

Proof:

Using (22) and (23), we can derive

$$C_{k,l} = C_{k,l-L} \tag{25}$$

In other words, the time-domain sequence possesses zero-autocorrelation property.

From (22) we have

$$g_k = \sqrt{L} c_k \tag{26}$$

Because  $g$  is a CAZAC sequence, we have

$$g_k g_{k-L}^* = 0 \tag{27}$$

Rewriting (27) in matrix form, we have



$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \quad (28)$$

Solving (28), we obtain

$$\begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \quad (29)$$

In other words, the sequence in time domain is also a CAZAC sequence.

Q.E.D.

From Claim 1 and Claim 2 we can observe that the CAZAC characteristics is preserved both in time and frequency domain.

Let  $x[n]$  be a time-domain waveform of length  $2L$  at Nyquist rate. Its frequency spectral components can be computed using (26), or

$$X[k] = \sum_{n=0}^{2L-1} x[n] e^{-j2\pi kn/2L} \quad (30)$$

where  $F$  is the Fourier transform matrix of dimension  $2L \times 2L$  and  $X_{low}$  and  $X_{high}$  are lower and upper portions of the frequency spectrum. When subsampling the waveform at symbol rate (half of the Nyquist rate), we introduces spectrum folding on the frequency domain. Let

$x_e[n]$  be the subsampled sequence of even-numbered samples and  $x_o[n]$  the odd-numbered one. Define  $S$  to be the matrix operation that rearranges matrix columns into even and odd columns, or

$$S = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (31)$$

We can see that

$$\begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \quad (32)$$

When simplified, we get

$$\begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \quad (33)$$

$$\frac{1}{\sqrt{2}} \left( \frac{E_{\text{even}} + E_{\text{odd}}}{2} \right) \quad (34)$$

where  $E_{\text{even}}$  and  $E_{\text{odd}}$  are frequency spectral components of the even and odd sample sequences, and

We can easily derive from (33) and (34) the following spectrum folding relationships.

$$\frac{1}{\sqrt{2}} \left( \frac{E_{\text{even}} + E_{\text{odd}}}{2} \right) \quad (35)$$

$$\frac{1}{\sqrt{2}} \left( \frac{E_{\text{even}} - E_{\text{odd}}}{2} \right) \quad (36)$$

Equation (35) and (36) sums up the spectral folding phenomenon of waveform subsampling.

## 5. References

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