MMSE Calculation of Reference Equalizer Taps for TDECQ

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1 Introduction

This document outlines the Minimum Mean-Squared Error (MMSE) formulation for determining the reference equalizer coefficients for Clause 180.9.6 TDECQ calculations. By employing a direct Wiener-Hopf matrix solution, this approach eliminates the need for computationally intensive brute-force equalizer coefficient searches.

1.1 Alphabet Definition and Scope Limitation

- **Sequence Alphabet Definition:** The ideal transmitted SSPRQ target sequence $x(n)$ is mapped utilizing the standard symmetric, zero-mean PAM4 integer alphabet $\{-3, -1, 1, 3\}$.

- **Scope Limitation regarding Tap Bounds:** This mathematical derivation represents an unconstrained global optimization baseline. The strict tap coefficient boundary limits outlined in Table 180-16 are not enforced in this derivation. If the unconstrained solution yields coefficients exceeding the tap limits, the final tap values must instead be determined using a bounded, constrained optimization framework such as quadratic programming.

The reference receiver architecture uses a Decision-Feedback Equalizer (DFE) with 15 feedforward taps and 1 feedback tap. Assuming correct decisions are fed back under a known test pattern sequence, the feedback path uses the ideal transmitted symbols. The captured waveform samples natively incorporate deterministic inter-symbol interference (ISI) and signal-dependent transmitter noise, while background Additive White Gaussian Noise (AWGN) is shaped by a 4-pole Bessel-Thomson filter.

2 System Model and Notation

Following the notation of IEEE Draft P802.3dj/D3.1, the total continuous-time input signal to the sampler is expressed as:

$$z(t) = r(t) + \eta(t) \tag{1}$$

where $r(t)$ is the received optical signal captured by the sampling scope—which has already undergone filtering by the 4-pole Bessel-Thomson filter and naturally includes any transmitter noise—and $\eta(t)$ represents the filtered background Gaussian noise.

The discrete-time signal sampled at the symbol rate with a sampling phase ϕ_0 is given by:

$$z_n = z(nT + \phi_0) \tag{2}$$

The reference equalizer computes its output y_n using 15 feedforward taps $w(a+i)$ for $i \in \{0, 1, \ldots, 14\}$ and a single feedback tap $b(1)$:

$$y_n = \sum_{i=0}^{14} w(a+i) z_{n-a-i} - b(1)x(n-1) \tag{3}$$

where $x(n)$ denotes the ideal transmitted SSPRQ symbol sequence, and $a \in \{-3, -2, -1, 0\}$ denotes the number of pre-cursor taps.

We establish a compact matrix formulation by defining the compound input vector $\mathbf{z}_n$ and the consolidated coefficient vector $\mathbf{w}$:

$$\mathbf{z}_n = \begin{bmatrix} z_n & z_{n-1} & \ldots & z_{n-14} & x(n-1) \end{bmatrix}^T \tag{4}$$

$$\mathbf{w} = \begin{bmatrix} w(a) & w(a+1) & \ldots & w(a+14) & -b(1) \end{bmatrix}^T \tag{5}$$

This allows the equalizer output expression to be written as a vector inner product:

$$y_n = \mathbf{w}^T \mathbf{z}_n \tag{6}$$

3 MMSE Formulation and Wiener-Hopf Solution

The error signal e_n relative to the aligned target symbol $x(n)$ is defined as:

$$e_n = y_n - x(n) = \mathbf{w}^T \mathbf{z}_n - x(n) \tag{7}$$

$$J(\mathbf{w}) = E\left[e_n^2\right] = E\left[\left(\mathbf{w}^T \mathbf{z}_n - x(n)\right)^2\right] \tag{8}$$

Expanding the expectation yields:

$$J(\mathbf{w}) = \mathbf{w}^T E\left[\mathbf{z}_n \mathbf{z}_n^T\right] \mathbf{w} - 2\mathbf{w}^T E\left[\mathbf{z}_n x(n)\right] + E\left[x(n)^2\right] \tag{9}$$

Defining the input autocorrelation matrix $\mathbf{R}_{zz} = E\left[\mathbf{z}_n \mathbf{z}_n^T\right]$ and the cross-correlation vector $\mathbf{p}_{zx} = E\left[\mathbf{z}_n x(n)\right]$, the objective function simplifies to:

$$J(\mathbf{w}) = \mathbf{w}^T \mathbf{R}_{zz} \mathbf{w} - 2\mathbf{w}^T \mathbf{p}_{zx} + E\left[x(n)^2\right] \tag{10}$$

Taking the gradient with respect to the tap vector $\mathbf{w}$ and setting it to zero minimizes the cost function:

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = 2\mathbf{R}_{zz}\mathbf{w} - 2\mathbf{p}_{zx} = \mathbf{0} \tag{11}$$

This results in the classic Wiener-Hopf matrix equation:

$$\mathbf{R}_{zz}\mathbf{w}_{\text{opt}} = \mathbf{p}_{zx} \implies \mathbf{w}_{\text{opt}} = \mathbf{R}zz^{-1}\mathbf{p}_{zx} \tag{12}$$

4 Partitioned Matrix Structure and Sequence Constraints

The global (16×16) correlation matrix $\mathbf{R}_{zz}$ partitions into sub-blocks isolating the feedforward and feedback dimensions:

$$\mathbf{R}_{zz} = \left[\begin{array}{c|c} \mathbf{R}_{zz,\mathrm{f}} & \mathbf{R}_{zx,\mathrm{fb}} \\ \hline \mathbf{R}_{xz,\mathrm{fb}} & R_{xx} \end{array}\right] \tag{13}$$

4.1 Composition of the Feedforward Sub-block $\mathbf{R}_{zz,\mathbf{f}}$

The 15×15 sub-matrix $\mathbf{R}_{zz,\mathrm{f}}$ captures the combined statistics of the captured signal and background noise. Because $r(t)$ and $\eta(t)$ are mutually uncorrelated, the matrix splits into:

$$\mathbf{R}_{zz,\mathrm{f}} = \mathbf{R}_{rr} + \mathbf{R}_{\eta\eta} \tag{14}$$

4.1.1 Computation of $\mathbf{R}_{rr}$ from Captured Samples

Because $r(t)$ corresponds to the actual captured noisy waveform samples from the oscilloscope, it inherently includes any transmitter noise contributions alongside the deterministic ISI profile. For a periodic SSPRQ test pattern of length $N = 65,535$, the indices $j, k \in \{0, 1, \ldots, 14\}$ of $\mathbf{R}_{rr}$ are evaluated directly via time-averaging:

$$[\mathbf{R}_{rr}]_{j,k} = \frac{1}{N} \sum_{n=1}^{N} r(nT - jT + \phi_0) r(nT - kT + \phi_0) \tag{15}$$

4.1.2 Derivation and Numerical Baseline of Background Noise Matrix $\mathbf{R}_{\eta\eta}$

The background noise matrix $\mathbf{R}_{\eta\eta}$ is a symmetric Toeplitz matrix uniquely determined by its first row vector $\mathbf{r}_{\eta\eta} = [R_{\eta\eta}(0), R_{\eta\eta}(1), \ldots, R_{\eta\eta}(14)]$. The continuous-time transfer function of a fourth-order low-pass Bessel-Thomson filter is defined by:

$$H_{\mathrm{BT}}(s) = \frac{105}{s^4 + 10s^3 + 45s^2 + 105s + 105} \tag{16}$$

To evaluate the normalization matching the specified 53.125 GHz -3 dB electrical power frequency cutoff at a symbol rate of 106.25 GBaud, the frequency scale factor maps via $s = j2\pi f \cdot \tau_0$ with $\tau_0 = \frac{2.113915}{2\pi \cdot 53.125 \times 10^9}$.

The continuous power spectral density of the filtered Gaussian process is proportional to $|H_{\mathrm{BT}}(f)|^2$. Applying the Wiener-Khinchin theorem, the autocorrelation function at a continuous time lag τ is found by taking the inverse Fourier transform:

$$R_{\eta\eta}(\tau) = \sigma_G^2 \cdot \frac{\int_{\infty}^{\infty} |H_{\mathrm{BT}}(f)|^2 e^{j2\pi f \tau} df}{\int_{-\infty}^{\infty} |H_{\mathrm{BT}}(f)|^2 df} \tag{17}$$

Sampling this correlation profile at uniform discrete symbol intervals $\tau = mT$, where $T = \frac{1}{106.25 \text{ GBaud}} \approx 9.41176$ ps, isolates the integer lag values $m = |j - k|$. Setting the total background noise power on-diagonal to $R_{\eta\eta}(0) = \sigma_G^2$, numerical integration of the normalized frequency domain structure yields the following unique row coefficients:

$$\mathbf{r}_{\eta\eta} = \sigma_G^2 \left[\ 1.0000, \quad 0.1170, \quad -0.0537, \quad 0.0151, \quad -0.0033, \quad 0.0006, \quad -0.0001, \quad \mathbf{0}_{1\times 8}\ \right] \tag{18}$$

where any entry for element index lag $m \geq 7$ decays completely below 10^{-4} and is truncated to zero.

4.2 Deterministic SSPRQ Cross-Correlation and Feedback Elements

Because SSPRQ is not perfectly independent and identically distributed (i.i.d.), the remaining variables must be computed directly from the pattern's structural indices:

- The true sequence variance is computed via:

$$R_{xx} = \frac{1}{N} \sum_{n=1}^{N} x^2(n-1) \tag{19}$$

- The j-th entry of the 15×1 cross-correlation block $\mathbf{R}_{zx,\mathrm{fb}}$ captures the non-zero correlation between the past transmitted symbol and the captured filtered waveform:

$$[\mathbf{R}_{zx,\mathrm{fb}}]_j = \frac{1}{N} \sum_{n=1}^{N} r(nT - jT + \phi_0)x(n-1) \tag{20}$$

- The j-th entry of the 16×1 global cross-correlation vector $\mathbf{p}_{zx}$ targeting the symbol $x(n)$ evaluates to:

$$[\mathbf{p}_{zx}]_j = \begin{cases} \frac{1}{N} \sum_{n=1}^{N} r(nT - jT + \phi_0)x(n) & \text{for } 0 \leq j \leq 14 \\ \frac{1}{N} \sum_{n=1}^{N} x(n-1)x(n) & \text{for } j = 15 \end{cases} \tag{21}$$